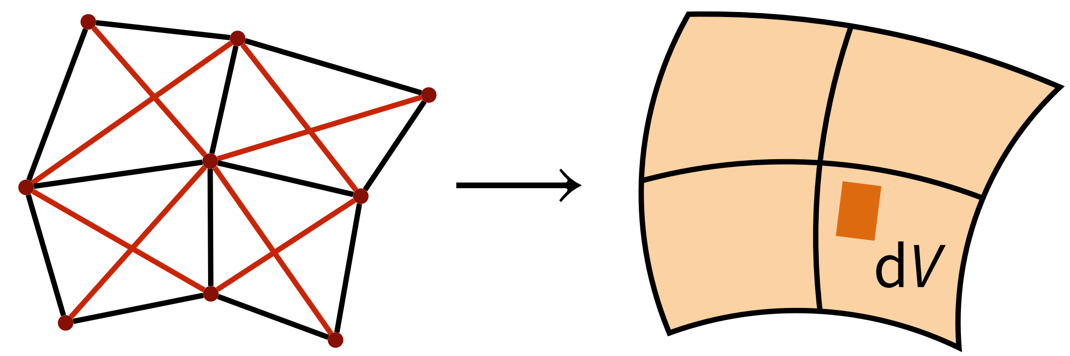
COL865: Special Topics in Computer Applications

Physics-Based Animation

12 — The Navier-Stokes equations for fluids

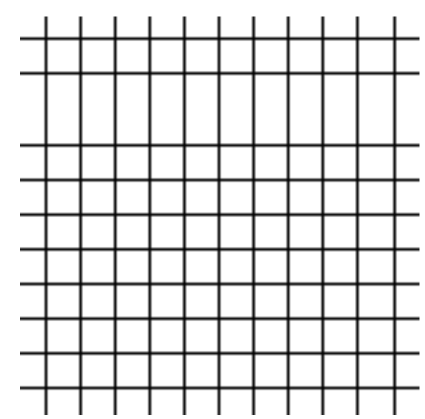
Review

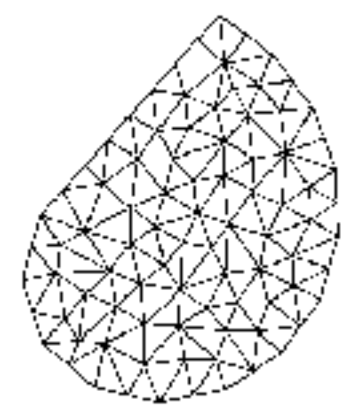
- Continuum models → PDEs involving both time and space

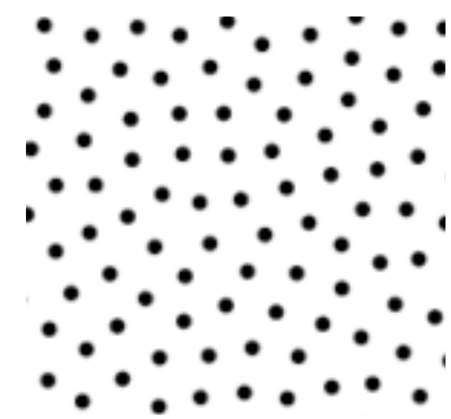


$$\rho\, \partial^2 \boldsymbol{\varphi}/\partial t^2 = (\ldots\, \partial \boldsymbol{\varphi}/\partial \mathbf{X}\, \ldots)$$

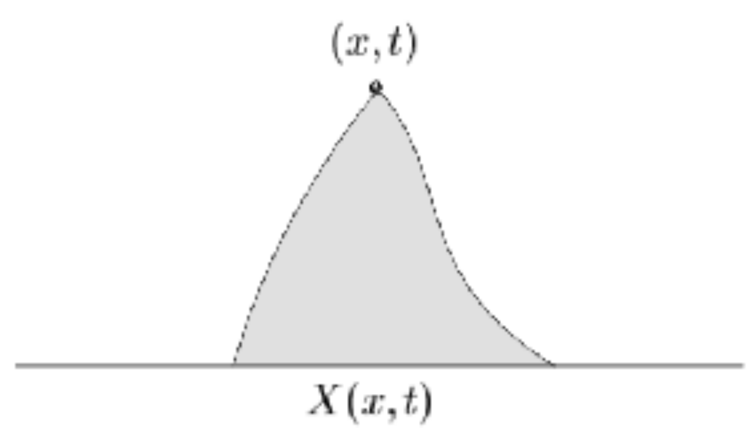
- Spatial discretizations

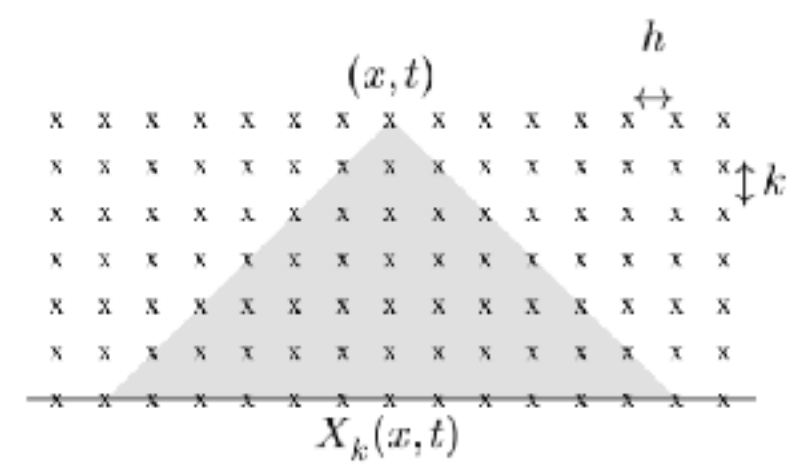






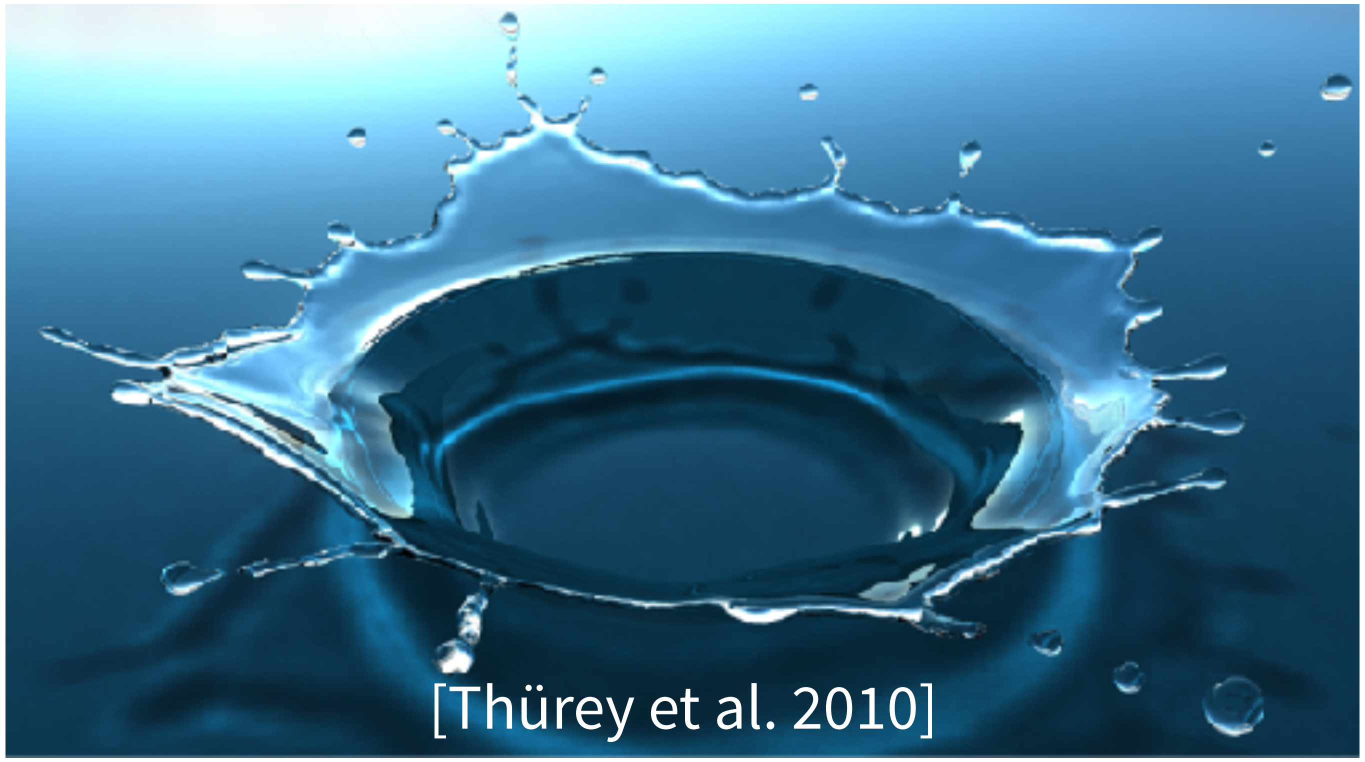
- Differentiation via finite differences
- The CFL condition



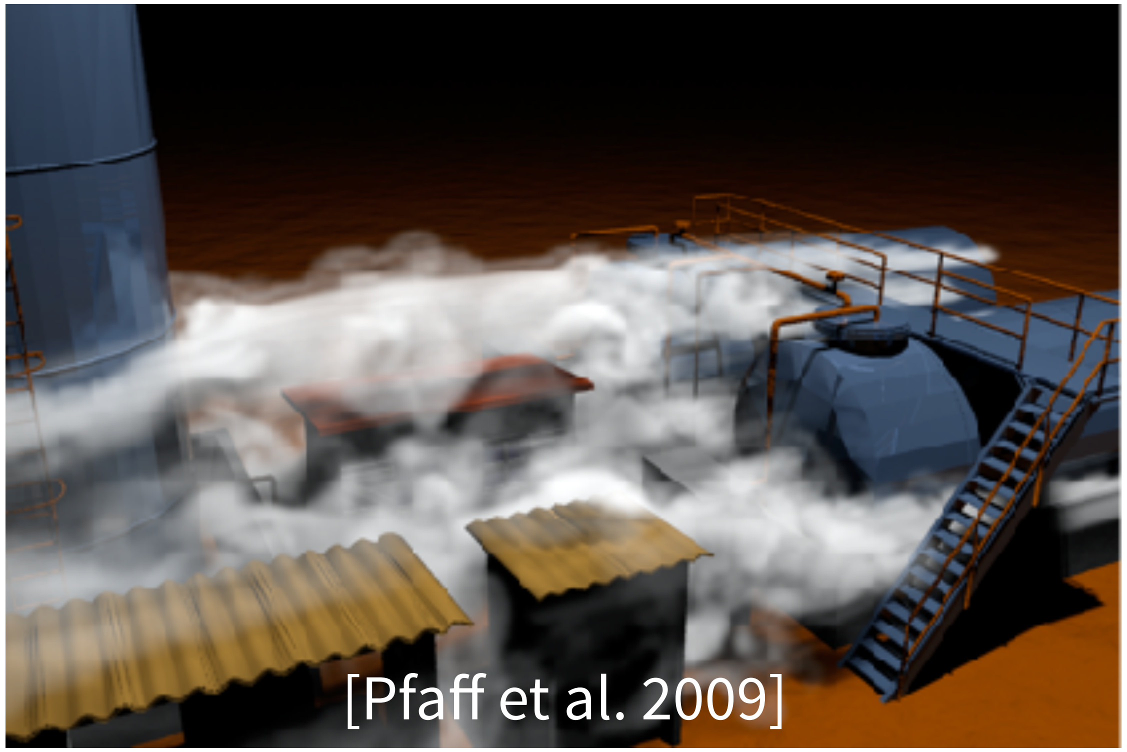


Fluids

[Fedkiw et al. 2001]



[Thürey et al. 2010]



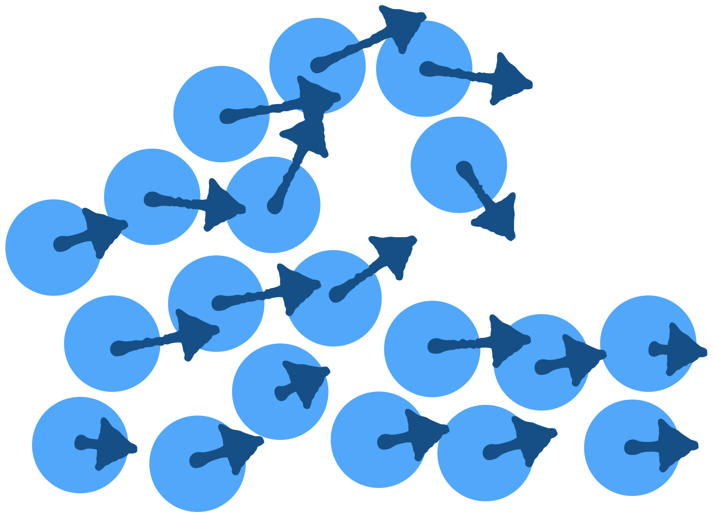
[Pfaff et al. 2009]

[Losasso et al. 2008]

Fluids

Continuous mass of particles moving freely

- ***Discrete***: position $\mathbf{x}_i$, velocity $\mathbf{u}_i$ for every particle
- ***Continuous***: velocity $\mathbf{u}(\mathbf{x})$ for every point $\mathbf{x}$



Eqs. of motion: ***Navier-Stokes equations***

PDE on velocity field $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$

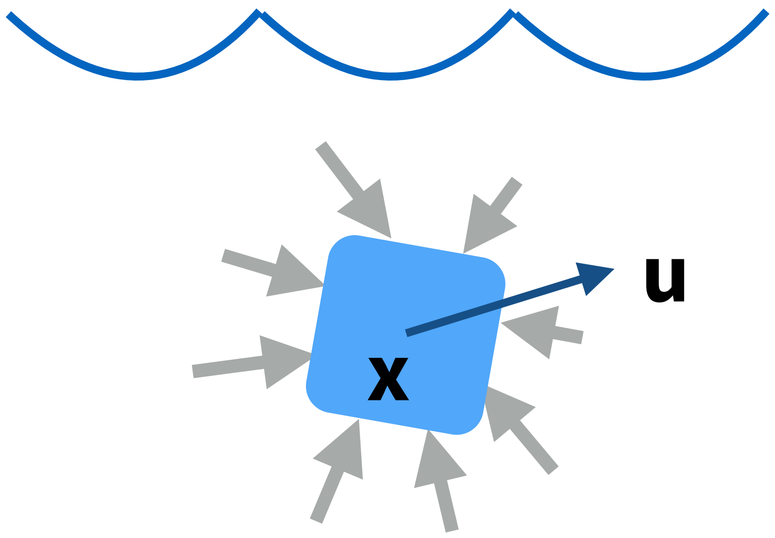
$$\partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \rho^{-1} (-\nabla p + \mu\, \nabla^2 \mathbf{u} + \mathbf{f}_{ext})$$

$$\nabla \cdot \mathbf{u} = 0$$

Deriving the Navier-Stokes equations

Equations of motion

Consider motion of infinitesimal "particle" of fluid



- Mass: $\mathrm{d}m = \rho\ \mathrm{dV}$
  Velocity: $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{u}(\mathbf{x}, t)$
  Force per unit volume: $\mathbf{f}$

Newton's second law:

$$\mathrm{d}m\ \mathrm{d}\mathbf{u}/\mathrm{d}t = \mathbf{f}\ \mathrm{d}V$$

$$\mathrm{d}\mathbf{u}/\mathrm{d}t = \rho^{-1}\ \mathbf{f}$$

This is not equal to $\partial\mathbf{u}/\partial t$, because the particle is moving!

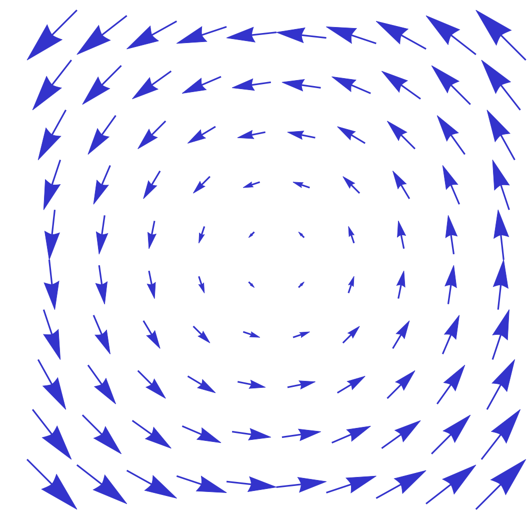
Eulerian and Lagrangian viewpoints

Acceleration of fluid particle is actually

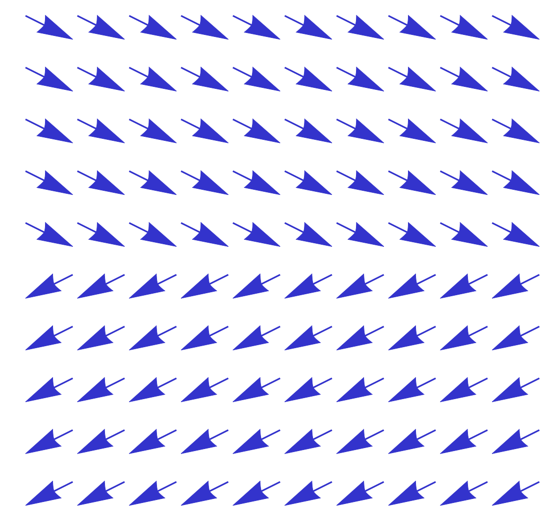
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(\mathbf{x}(t),t) = \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$$

$$= \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \mathbf{u}$$

- ***Eulerian*** view: rate of change at fixed point
- ***Lagrangian*** view: rate of change seen by particle moving with fluid



$\partial\mathbf{u}/\partial t = 0$, $\mathrm{d}\mathbf{u}/\mathrm{d}t \neq 0$



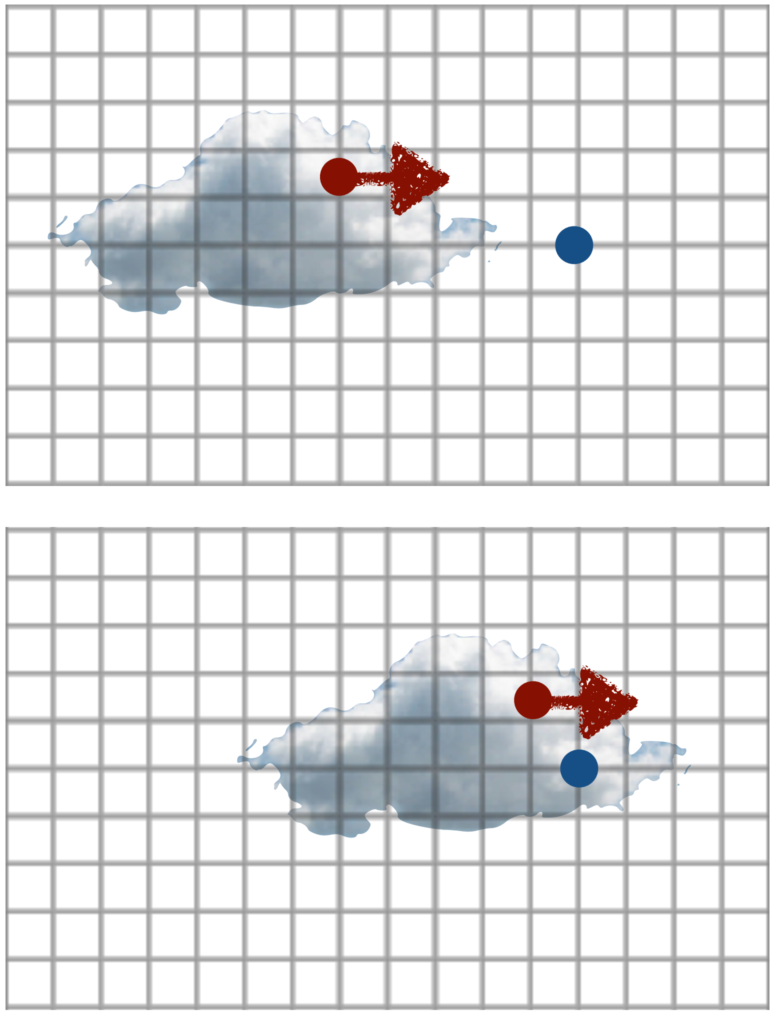
$\partial\mathbf{u}/\partial t \neq 0$, $\mathrm{d}\mathbf{u}/\mathrm{d}t = 0$

Advection

Consider a quantity c transported by fluid (e.g. smoke, temperature, colour, …)

$dc/dt = 0$ in Lagrangian frame

$\Rightarrow \partial c/\partial t + \mathbf{u} \cdot \nabla c = 0$: ***advection equation***



Lagrangian derivative
(or ***material derivative***):

$$D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$$

Fluid acceleration: $D\mathbf{u}/dt = \partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\, \mathbf{u}$
(velocity field advects itself!)

Navier-Stokes equations

$$\partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \rho^{-1}\, \mathbf{f}$$

What are the forces **f**?

- Body forces (e.g. gravity)
- Pressure
- Viscosity
- Surface tension
- Solid interaction

Forces

$$\partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \rho^{-1}\, \mathbf{f}$$

What are the forces **f**?

- Body forces (e.g. gravity = $\rho \mathbf{g}$)
- Pressure = $-\nabla p$
- Viscosity = $\mu\, \nabla^2 \mathbf{u}$
- Surface tension
- Solid interaction

} boundary conditions

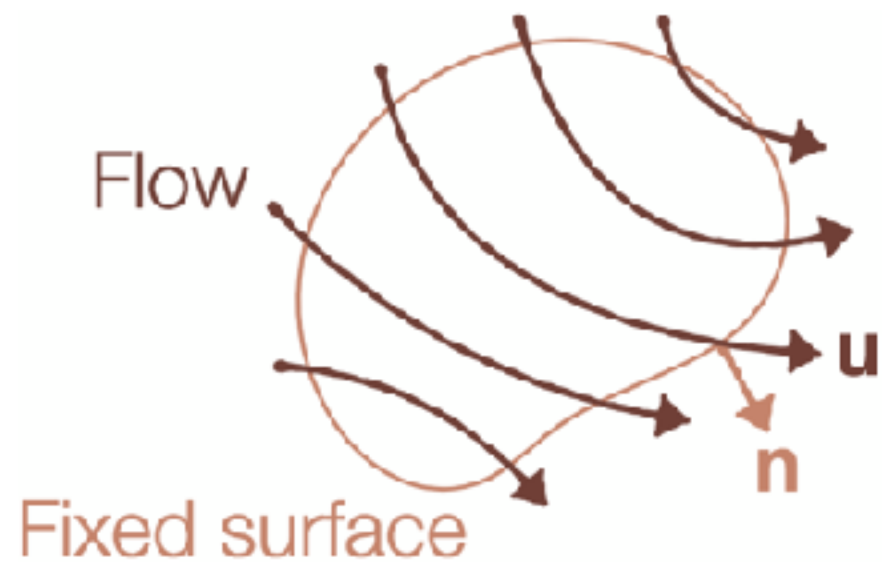
Pressure

- ***Real life***: pressure opposes compression → sound waves!
- ***Model***: treat fluid as ***incompressible***: ρ = const.
  Pressure acts as constraint force

Mass enclosed by surface = ρV

$$\text{Net mass flow out} = \oint\!\!\!\oint \rho \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}A$$

$$= \rho \iiint \nabla \cdot \mathbf{u} \, \mathrm{d}V$$



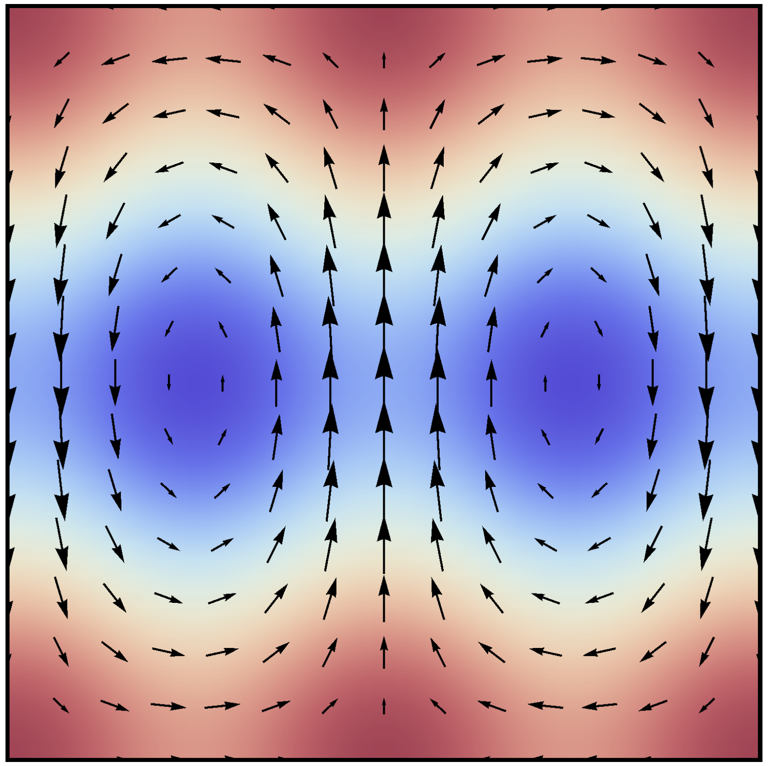
This should be zero for all possible surfaces ⇒ $\nabla \cdot \mathbf{u} = 0$

Pressure

$$\nabla \cdot \mathbf{u} = 0$$

Constraint maintained by ***pressure***:
scalar field $p : \mathbb{R}^3 \to \mathbb{R}$

- Force due to pressure $= -\nabla p$



Constrained dynamics viewpoint:

$$\mathbf{g}(\mathbf{x}) = \mathbf{0} \quad \to \quad \rho = \text{const}$$

$$\mathbf{J}\,\mathbf{v} = \mathbf{0} \quad \to \quad \nabla \cdot \mathbf{u} = 0$$

$$\mathbf{f}_\mathrm{c} = \mathbf{J}^\mathrm{T} \boldsymbol{\lambda} \quad \to \quad \mathbf{f}_\mathrm{p} = -\nabla p$$

The Navier-Stokes equations

material derivative — pressure — viscosity — body forces

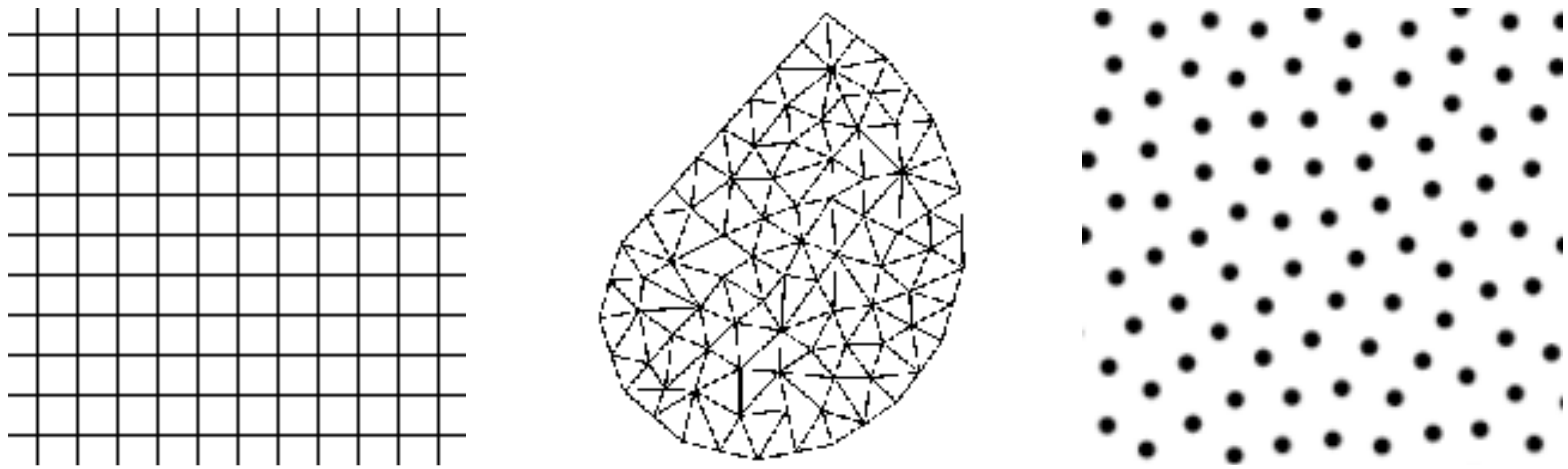
$$\partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \rho^{-1} \left(-\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}_{\text{ext}}\right)$$

$$\nabla \cdot \mathbf{u} = 0$$

incompressibility

Fluid simulation

Spatial discretization



- ***Grids***: Eulerian is easy with finite differences
- ***Particles***: Lagrangian is easy → ***smoothed particle hydrodynamics***
- ***Meshes***: Eulerian [Klingner et al. 2006, Chentanez et al. 2007, Batty et al. 2010] or Lagrangian [Misztal et al. 2010, Clausen et al. 2013]?

Grid-based fluids reading

- Bridson and Müller-Fischer, *Fluid Simulation for Computer Animation*, Ch. 1-4
  - ***Textbook***: Bridson, *Fluid Simulation for Computer Graphics*
- Stam, “Stable Fluids”, 1999

Time integration

assume $\rho = 1$

$$\partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \rho^{-1} \left(-\nabla p + \mu\, \nabla^2 \mathbf{u} + \mathbf{f}_{\text{ext}}\right)$$

$$\nabla \cdot \mathbf{u} = 0$$

Splitting: solve one term at a time over Δt each

1. Advection: $\partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = 0$
2. Body forces: $\partial \mathbf{u}/\partial t = \mathbf{f}_{\text{ext}}$
3. Viscosity: $\partial \mathbf{u}/\partial t = \nu\, \nabla^2 \mathbf{u}$
4. Pressure: $\partial \mathbf{u}/\partial t = -\nabla p$, $\nabla \cdot \mathbf{u} = 0$