COL865: Special Topics in Computer Applications

Physics-Based Animation

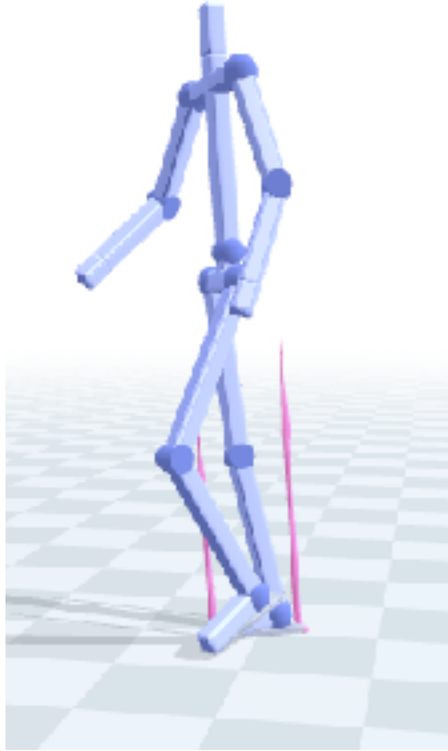
7 — Constrained dynamics

Constraints

A spring is a "stretchy" relationship between particles

What about non-stretchy relationships?

- Particle sliding along line
- Materials that don't stretch
- Bodies connected by joints



[Goldenthal et al. 2007]

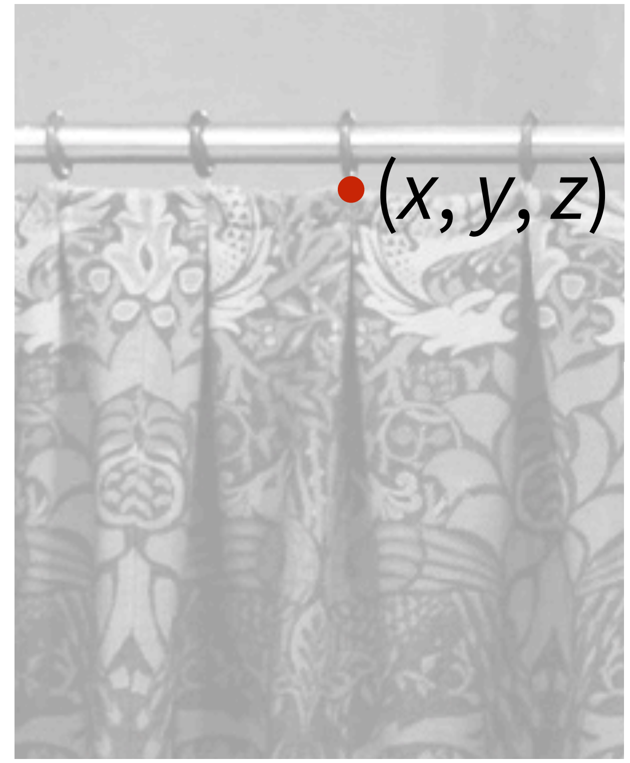
[Muico et al. 2009]

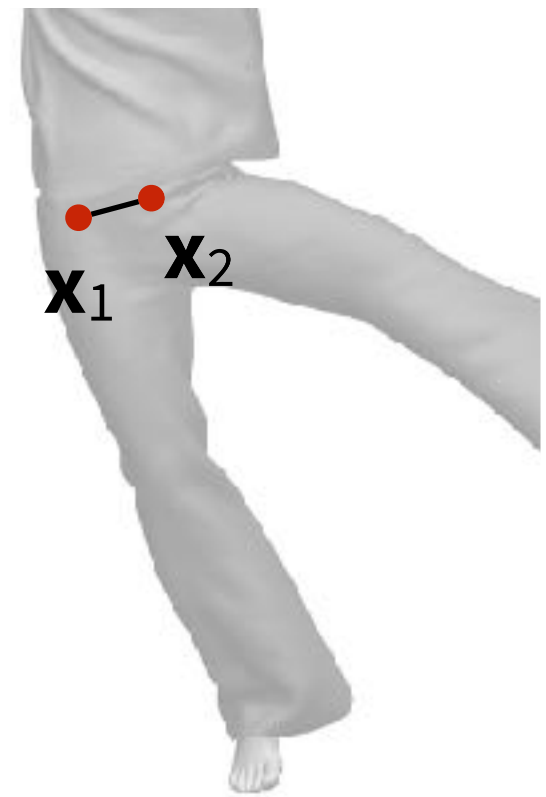
Constraints

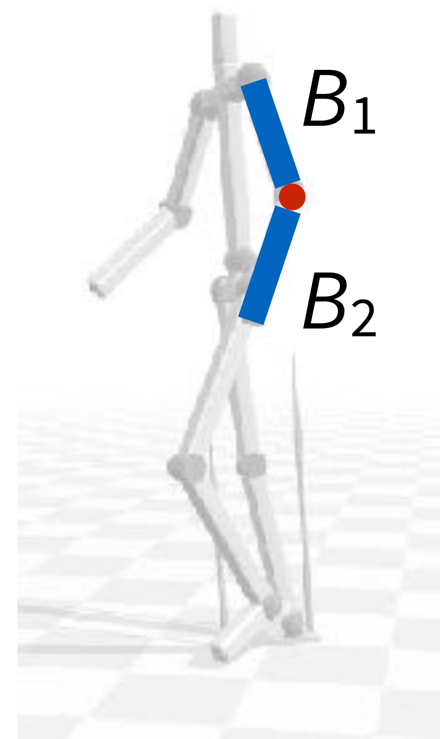
Many constraints can be modeled as equations

- Particle sliding along x-axis: $y = 0, z = 0$
- Inextensible material: $\|\mathbf{x}_1 - \mathbf{x}_2\| = \ell$
- Joint connecting bodies: $\mathbf{x}_1(B_1) = \mathbf{x}_2(B_2)$

General form: $\mathbf{g}(\mathbf{x}) = \mathbf{0}$





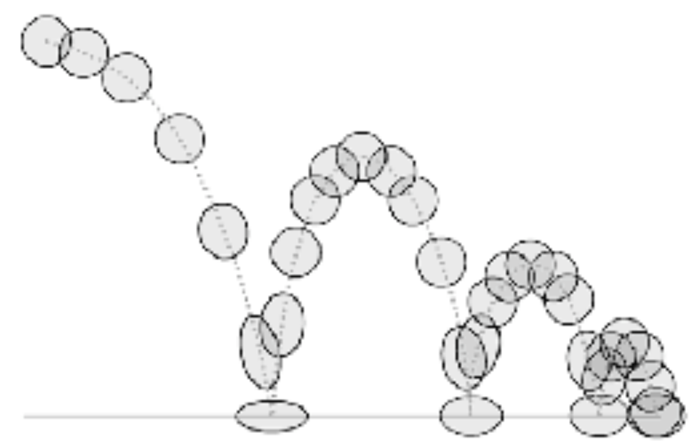


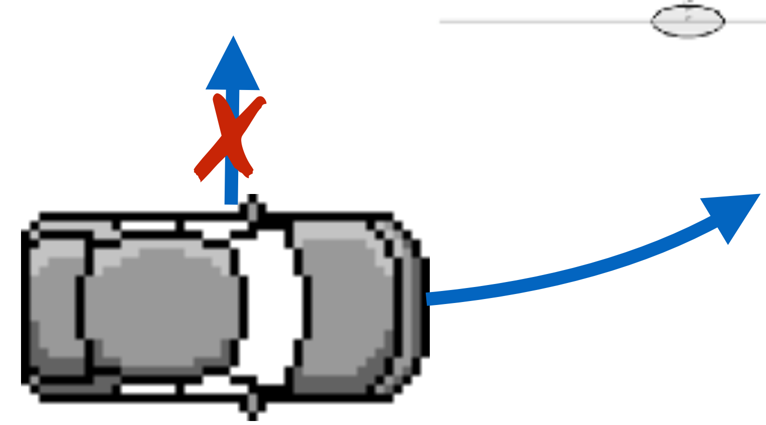
Types of constraints

Holonomic constraints: $\mathbf{g}(\mathbf{x}) = \mathbf{0}$

Nonholonomic constraints

- ***Inequality constraints***: e.g. $y \geq 0$
- ***Velocity constraints***:
  e.g. $\mathbf{v}_{\text{car}} \,||\, \mathbf{d}_{\text{car}}$





We'll discuss only holonomic constraints today

Stiff forces and constraints

Consider a spring attached to the origin:

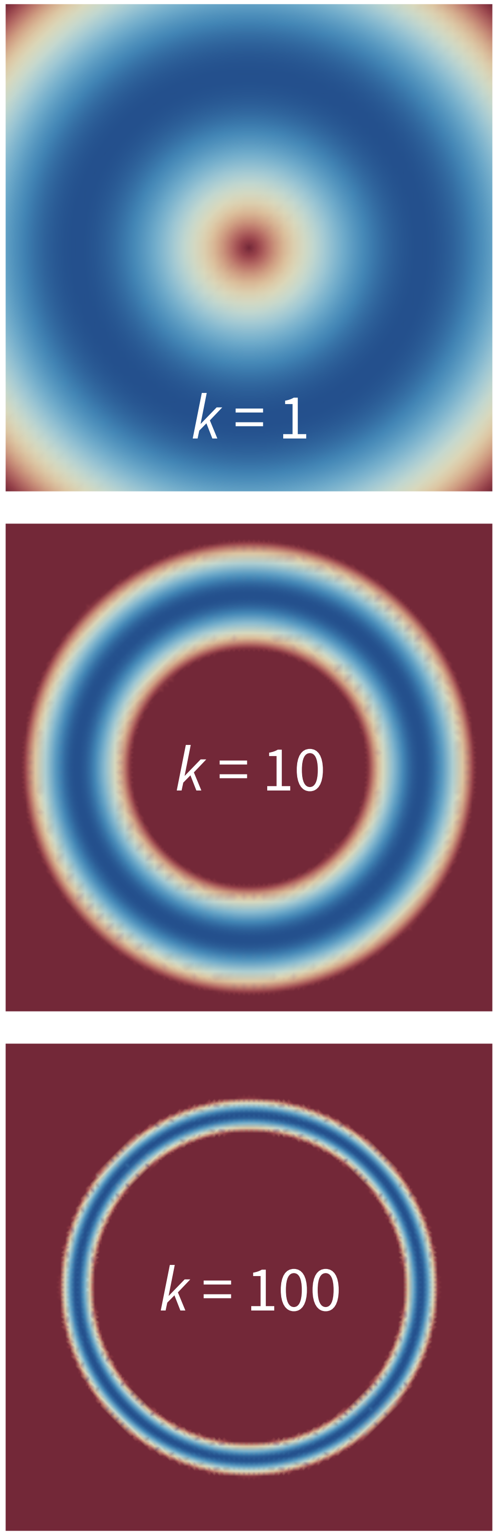
$$U(\mathbf{x}) = \tfrac{1}{2}\, k(\|\mathbf{x}\| - \ell)^2$$

In the limit as $k \to \infty$,

$$U(\mathbf{x}) = \begin{cases} 0 & \text{if } \|\mathbf{x}\| = \ell, \\ \infty & \text{otherwise.} \end{cases}$$

so $\|\mathbf{x}\|$ must be exactly ℓ:
the spring becomes an inextensible rod.

Constraints are a modeling simplification:
If we don't care about changes in a quantity (e.g. length of stiff spring), make it constant



Constrained dynamics

How to describe the motion of a system subject to constraints?

- **Reduced coordinates**
- **Penalty forces**
- **Differentiating the constraints**
- **Projection methods**

Recommended reading

- Witkin and Baraff, *Physically Based Modeling*, Ch. “Constrained Dynamics”
- Goldenthal et al., “Efficient Simulation of Inextensible Cloth”, 2007

Reduced coordinates

Reduced coordinates: pendulum

Replace “maximal coordinates” **x** with smaller set of coordinates which automatically satisfy the constraints

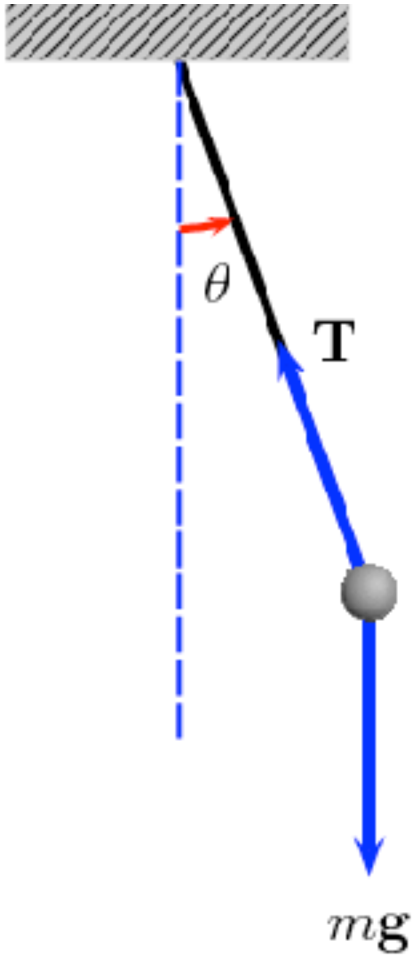
e.g. pendulum = particle + inextensible rod + gravity

Natural ***kinematic*** description:
angle θ, time derivative $\dot{\theta}$

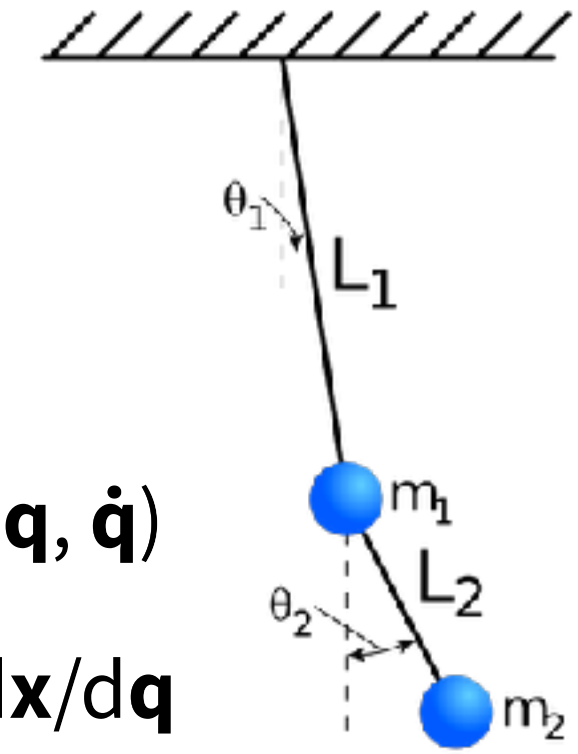
- When needed, get $\mathbf{x} = \mathbf{x}(\theta)$, $\mathbf{v} = (\mathrm{d}\mathbf{x}/\mathrm{d}\theta)\,\dot{\theta}$

But the ***dynamics*** becomes more complicated:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{\ell}\sin\theta = 0$$



Reduced coordinates

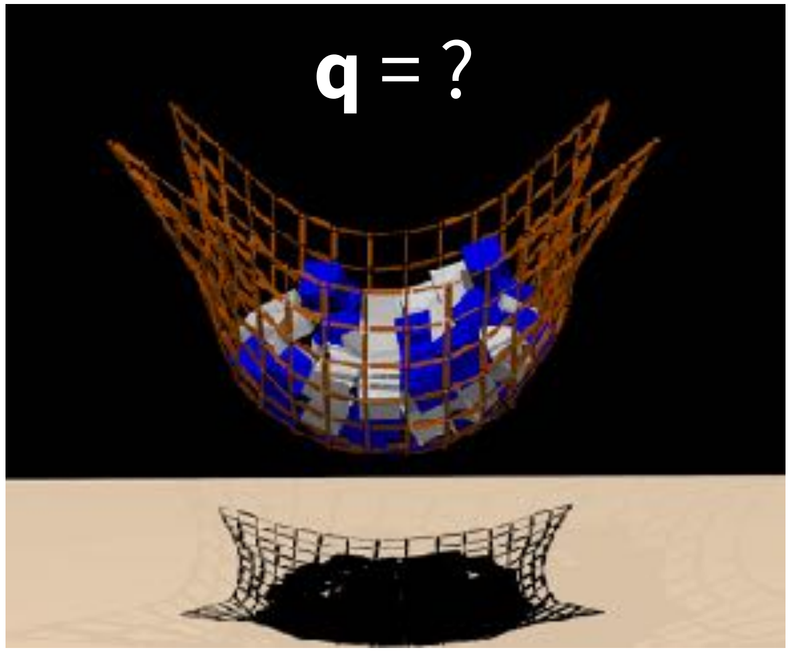


General formula: ***Lagrangian mechanics***

1. Choose ***generalized coordinates***, $\mathbf{q} \in \mathbb{R}^d$
2. Write down a formula for the kinetic energy $T(\mathbf{q}, \dot{\mathbf{q}})$
3. Define ***generalized forces***, $\mathbf{Q} = \mathbf{J}^\mathsf{T} \mathbf{f}$ where $\mathbf{J} = \mathrm{d}\mathbf{x}/\mathrm{d}\mathbf{q}$
4. Then equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}T}{\mathrm{d}\dot{\mathbf{q}}}\right) - \frac{\mathrm{d}T}{\mathrm{d}\mathbf{q}} = \mathbf{Q}$$

Reduced coordinates



[Weinstein et al. 2006]

Does not scale well for general systems with arbitrary constraints $\mathbf{g}(\mathbf{x}) = \mathbf{0}$.

Useful in some simple cases e.g. ***linear*** constraints (particle on plane/line/point):

- Choose basis of allowed directions $\mathbf{B}$, let $\mathbf{x} = \mathbf{B}\,\mathbf{q}$

$$\mathbf{B}^\mathsf{T}\,\mathbf{M}\,\mathbf{B}\;\mathrm{d}\dot{\mathbf{q}}/\mathrm{d}t = \mathbf{B}^\mathsf{T}\,\mathbf{f}$$

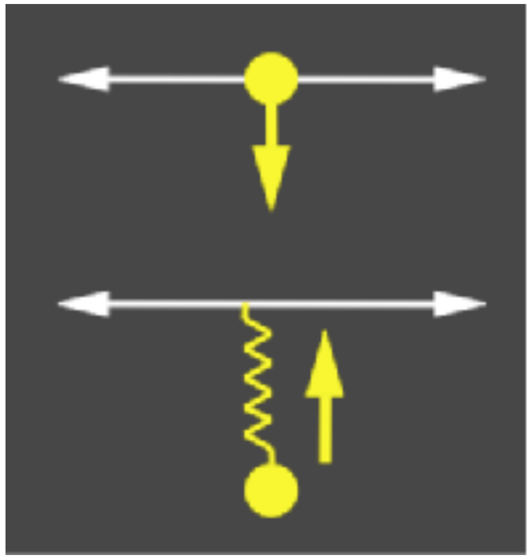
Penalty forces

Penalty forces

A constraint is like an infinitely steep potential. Replace it with a finitely steep one: a "***soft***" constraint

Inextensible rod → spring:

$$\|\mathbf{x}\| = \ell \quad \rightarrow \quad U(\mathbf{x}) = \tfrac{1}{2}\, k(\|\mathbf{x}\| - \ell)^2$$



[Witkin & Baraff]

In general, for a single constraint:

$$g(\mathbf{x}) = 0 \quad \rightarrow \quad U_c(\mathbf{x}) = \tfrac{1}{2}\, \alpha\, g(\mathbf{x})^2$$

Constraint force pushes system back towards $g(\mathbf{x}) = 0$:

$$\begin{aligned} \mathbf{f}_c(\mathbf{x}) &= -\nabla U_c(\mathbf{x}) \\ &= -\alpha\, g(\mathbf{x})\, \nabla g(\mathbf{x}) \end{aligned}$$

Penalty forces

$$g(\mathbf{x}) = 0$$
$$U(\mathbf{x}) = \tfrac{1}{2}\,\alpha\, g(\mathbf{x})^2$$
$$\mathbf{f}_c(\mathbf{x}) = -\alpha\, g(\mathbf{x})\, \nabla g(\mathbf{x})$$

Can also add a damping term $-\beta\, \dot{g}\, \nabla g$, where $\dot{g} = \nabla g \cdot \mathbf{v}$

Limitations:

- If α is too small, constraint will be violated a lot
- If α is large, simulation may explode!
  (or with an implicit method, matrix will be ill-conditioned)

Differentiating the constraints

Valid positions, velocities, accelerations

Let's rigorously derive the equations of motion.

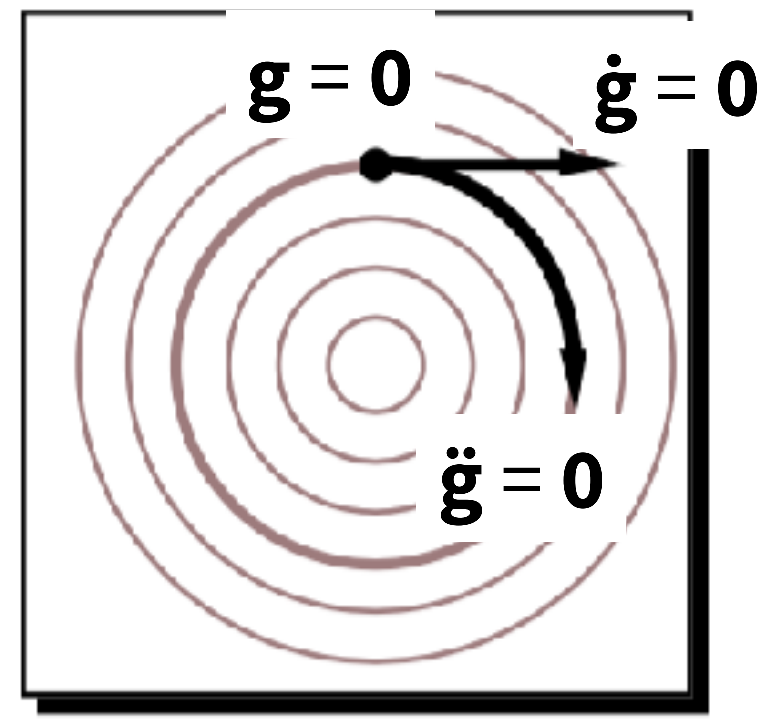
Given

$$\mathbf{g}(\mathbf{x}) = \mathbf{0},$$

differentiate with respect to time:

$$\dot{\mathbf{g}} = \mathbf{J}\,\mathbf{v} = \mathbf{0},$$
$$\ddot{\mathbf{g}} = \mathbf{J}\,\dot{\mathbf{v}} + \dot{\mathbf{J}}\,\mathbf{v} = \mathbf{0}$$

$\ddot{\mathbf{g}} = \mathbf{0}$ implies a condition on $\dot{\mathbf{v}}$. There has to be a constraint force $\mathbf{f}_c$ to satisfy it.



[Witkin and Baraff]

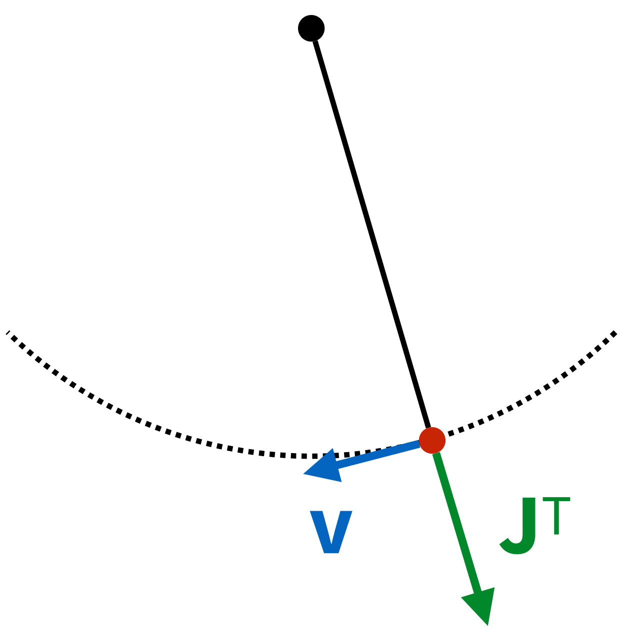
Pendulum: $\dot{\mathbf{J}}\,\mathbf{v} = \|\mathbf{v}\|^2/\ell$ = centrifugal acceleration

Constraint force

$$\mathbf{J}\,\mathbf{v} = \mathbf{0}$$
$$\mathbf{J}\,\dot{\mathbf{v}} + \dot{\mathbf{J}}\,\mathbf{v} = \mathbf{0}$$

Constraint is passive so $\mathbf{f}_c$ must do no work

$$\mathbf{f}_c \cdot \mathbf{v} = 0 \text{ for } \textit{all valid } \mathbf{v}$$
$$\Rightarrow\ \mathbf{f}_c = \mathbf{J}^T \boldsymbol{\lambda} \text{ for some } \boldsymbol{\lambda}$$



Pendulum: $\mathbf{v}$ is tangential, $\mathbf{J}^T$, $\mathbf{f}_c$ are radial

If n particles, m constraints, $\mathbf{g} : \mathbb{R}^{3n} \to \mathbb{R}^m$, $\mathbf{J} : 3m \times n$, $\boldsymbol{\lambda} \in \mathbb{R}^m$

Constrained equations of motion

$$\dot{\mathbf{x}} = \mathbf{v}$$

$$\dot{\mathbf{v}} = \mathbf{M}^{-1} (\mathbf{f} + \mathbf{J}^{\mathsf{T}} \boldsymbol{\lambda})$$

$$\mathbf{g}(\mathbf{x}) = \mathbf{0}$$

This is a ***differential-algebraic equation***

$\boldsymbol{\lambda}$ is unknown, but fixed by the requirement that $\mathbf{g}(\mathbf{x}) = \mathbf{0}$. Compute it explicitly using time derivatives of $\mathbf{g}$:

$$\mathbf{J}\,\dot{\mathbf{v}} + \dot{\mathbf{J}}\,\mathbf{v} = \mathbf{0}$$

$$\Rightarrow\ \mathbf{J}\,\mathbf{M}^{-1}\,\mathbf{J}^{\mathsf{T}}\,\boldsymbol{\lambda} = -(\mathbf{J}\,\mathbf{M}^{-1}\,\mathbf{f} + \dot{\mathbf{J}}\,\mathbf{v})$$

Solve for $\boldsymbol{\lambda}$, done!

Constrained equations of motion

$$\dot{\mathbf{x}} = \mathbf{v}$$

$$\dot{\mathbf{v}} = \mathbf{M}^{-1} (\mathbf{f} + \mathbf{J}^T \boldsymbol{\lambda})$$

$$\text{where } (\mathbf{J}\, \mathbf{M}^{-1} \mathbf{J}^T)\, \boldsymbol{\lambda} = -(\mathbf{J}\, \mathbf{M}^{-1} \mathbf{f} + \dot{\mathbf{J}}\, \mathbf{v})$$

Interpretation:

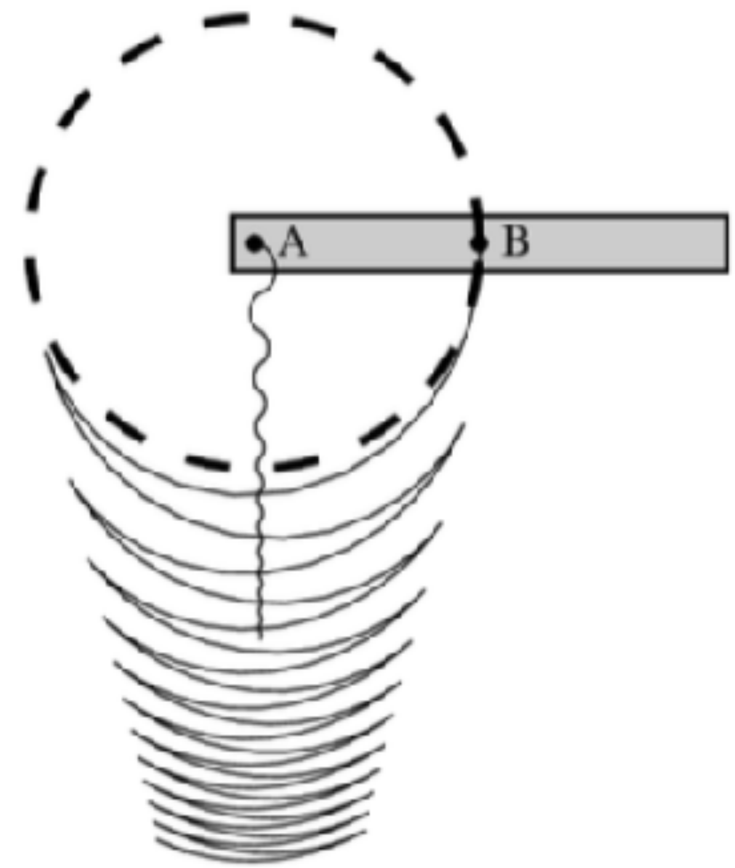
- $\mathbf{J}^T \boldsymbol{\lambda}$ = constraint force, $\mathbf{M}^{-1} \mathbf{J}^T \boldsymbol{\lambda}$ = resulting acceleration, $\mathbf{J}\, \mathbf{M}^{-1} \mathbf{J}^T \boldsymbol{\lambda}$ = acceleration in constrained direction
- This should exactly cancel out acceleration due to **f** and **v**

Pendulum: $\boldsymbol{\lambda} = -(\mathbf{f}_r + m\|\mathbf{v}\|^2/\ell)$ = centripetal force due to rod

Constrained equations of motion

Problem: We have enforced $\ddot{\mathbf{g}} = \mathbf{0}$ instead of $\mathbf{g} = \mathbf{0}$. Numerical error leads to drift!

Baumgarte stabilization:
Replace $\ddot{\mathbf{g}} = \mathbf{0}$ with $\ddot{\mathbf{g}} = -\alpha\, \mathbf{g} - \beta\, \dot{\mathbf{g}}$ to push the system back towards the constraint



[Cline and Pai 2003]