Amitabha Bagchi
Scribe: Arpit Nanda

Lecture 9: Supercritical phase: The infinite cluster is unique

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We now move to the supercritical phase. We argued earlier that when $p > p_c$, clusters of infinite size exists. In this lecture we will show that the number of infinite clusters in the supercritical phase is exactly equal to 1 i.e. the infinite cluster is unique in the supercritical phase.

9.1 The infinite cluster is unique

Theorem 9.1 *For any p s.t. $\theta(p) > 0$,*

$$P_p(\textit{There is exactly one infinite open cluster}) = 1$$

Proof. Let us denote the number of infinite clusters by N. N is a random variable whose value depends on p. We will show that if $\theta(p) > 0$, $\mathrm{P}_p(N = 1) = 1$. Before that we define some notation. Given a finite set of vertices B, E_B being the set of edges that connect vertices of B, we define the following random variables

- $N_B(0)$: No of infinite clusters in the lattice when all edges in E_B are closed.
- $N_B(1)$: No of infinite clusters in the lattice when all edges in E_B are open.

Since closing any edge set can potentially break a cluster into multiple clusters and deliberately opening edges can possibly join multiple clusters, we can deduce that $N_B(0) \geq N_B(1)$.

Let us also denote by M_B the number of infinite open clusters which intersect B. Clearly, as B grows towards $\mathbb{Z}^d$, M_B will tend towards N i.e. as $B \uparrow \mathbb{Z}^d$ then $M_B \to N$.

Our proof depends centrally on the fact, presented here without proof, that any translation invariant function on the product space Ω is almost surely constant i.e.

$$\exists k \in \{0, 1, 2 \ldots\} \cup \infty \;\text{ s.t. } \mathrm{P}_p(N = k) = 1.$$

And so, to prove the theorem, we would need to rule out all values of k except 1. We do this in two parts, first showing that k cannot take any finite value that is 2 or greater and then we will separately argue that k cannot be ∞.

Lemma 9.2 *For $0 < p < 1$, and for k such that $2 \leq k < \infty$,*

$$P_p(N = k) = 0.$$

Proof of Lemma 9.2. Let us assume that $N = k$ for some $2 \leq k < \infty$. In order to show that this event has probability 0, we consider some finite set of vertices B, with edge set E_B.

Let us assume that B intersects more than one of the k infinite components that exist. If this is the case then $N_B(0) \geq k$ since closing all the edges of E_B might cause one of the infinite components to break up into multiple infinite components (or it might not). But we can definitely say that $N_B(1) < k$ since opening all the edges of E_B will cause the multiple infinite components B intersects to merge into one.

Hence, we can say that

$$\begin{aligned} \mathrm{P}_p(N \geq k) &\geq \mathrm{P}_p(N_B(0) \geq k) \cdot \mathrm{P}_p(\text{All of } E_B \text{ is closed}) \\ &= (1-p)^{|E_B|} \end{aligned}$$

since by our assumption $\mathrm{P}_p(N_B(0) \geq k) = 1$. Similarly we have that

$$\begin{aligned} \mathrm{P}_p(N < k) &\geq \mathrm{P}_p(N_B(1) < k) \cdot \mathrm{P}_p(\text{All of } E_B \text{ is open}) \\ &= p^{|E_B|} \end{aligned}$$

Hence, if B intersects more than one infinite component then we have a contradiction to the fact that N is a.s. constant since there is non-zero probability associated with values of N both greater than equal to k and strictly less than k. And so, for any finite B it must be the case that $\mathrm{P}_p(M_B \geq 2) = 0$.

Now suppose we take $B = S(n)$. As $n \to \infty$ $S(n), \to \mathbb{Z}^d$ and so $M_B \to N$. But $\mathrm{P}_p(M_B \geq 2) = 0$ for all finite B and so the result follows. ∎

It remains to rule out the possibility that N can take the value ∞.

Lemma 9.3 *If $0 < p < 1$, then*

$$P_p(N = \infty) = 0.$$

Proof of Lemma 9.3. We will prove this by contradiction. Our strategy will be to show that if we assume that N takes the value ∞, in fact, if N is greater than 3, it is possible to come up with a mapping of the interior of the box $B(n)$ into its boundary $\partial B(n)$. This is not possible since $|\partial B(n)| = \theta(n^{d-1})$ and $|B(n)| = \theta(n^d)$ and so our original assumption (that $N = \infty$) is wrong.

In order to do this we study vertices with the following property.

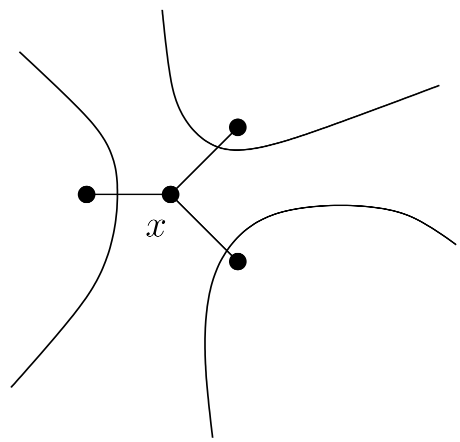

Figure 1: x is a trifurcation.

Definition 9.4 *A vertex x is called a* trifurcation *if*

- *x lies in an infinite open cluster,*
- *x has only 3 open edges incident to it, and*
- *removing x and its open edges splits the infinite cluster into exactly 3 infinite clusters.*

We denote by T_x the event that x is a trifurcation. I_{T_x} is the indicator function associated with T_x. Clearly the expected number of trifurcations inside the box $B(n)$ are given by

$$\mathrm{E}_p\left(\sum_{x \in B(n)} I_{T_x}\right) = |B(n)|\mathrm{P}_p(T_x). \tag{1}$$

We will now show that $\mathrm{P}_p(T_x)$ is a constant. If we are able to show this then it will follow from (1) that the expected number of trifurcations inside $B(n)$ are linear in the size of $B(n)$ i.e. are $\theta(n^d)$.

Since T_x is a translation invariant event, we will focus on T_0, the event that the origin is a trifurcation. Our strategy to lower bound $\mathrm{P}_p(T_0)$ will be in two stages. We will first show that there is an n^* such that $S(n*)$ intersects 3 infinite clusters with probability at least 1/2. Then we will close all the edges in $S(n^*)$ except the edges corresponding to three edge disjoint paths leading from the origin to points on $\partial S(n^*)$ which intersect the three infinite clusters we found in the earlier step. This is one specific way of making the origin a trifurcation and the probability associated with this will give us a lower bound on the probability that 0 is a trifurcation.

For the first step of our strategy we need some notation. Let us denote the number of infinite clusters intersecting a finite set B when its edge set E_B is closed by $M_B(0)$ and the number of infinite clusters intersecting B when E_B is open by $M_B(1)$. Additionally we also define M_B to be the number of infinite clusters intersecting the vertices of the set B. Clearly the number of infinite clusters intersecting B will be at most the number of infinite clusters intersecting B when E_B is closed, since closing the edges of E_B may cause some cluster or clusters to split into multiple clusters. In other words

$$\{M_B \geq k\} \Rightarrow \{M_B(0) \geq k\}. \tag{2}$$

Setting $B = S(n)$ and $k = 3$ in (2) we get

$$\mathrm{P}_p(M_{S(n)}(0) \geq 3) \geq \mathrm{P}_p(M_{S(n)} \geq 3).$$

Also, since $\mathrm{P}_p(M_{S(n)} \geq 3) \to \mathrm{P}_p(N \geq 3)$ as $n \to \infty$ and since we have assumed that $\mathrm{P}_p(N = \infty) = 1$,

$$\lim_{n\to\infty} \mathrm{P}_p(M_{S(n)}(0) \geq 3) = 1.$$

Hence there must be some n^* such that $\mathrm{P}_p(M_{S_{n^*}}(0) \geq 3) \geq \frac{1}{2}$. This is the n^* we had referred to earlier, and having found this we are done with the first step of our strategy to lower bound $\mathrm{P}_p(T_0)$.

If $S(n^*)$ does intersect 3 infinite clusters, let us call them C_1, C_2 and C_3, there must be three vertices x_1, x_2, x_3 such that $x_i \in \partial S(n^*) \cap C_i, i = 1, 2, 3$, and x_i is contained in an infinite cluster even when all the edges of $E_{S(n^*)}$ are closed (see Figure 2).

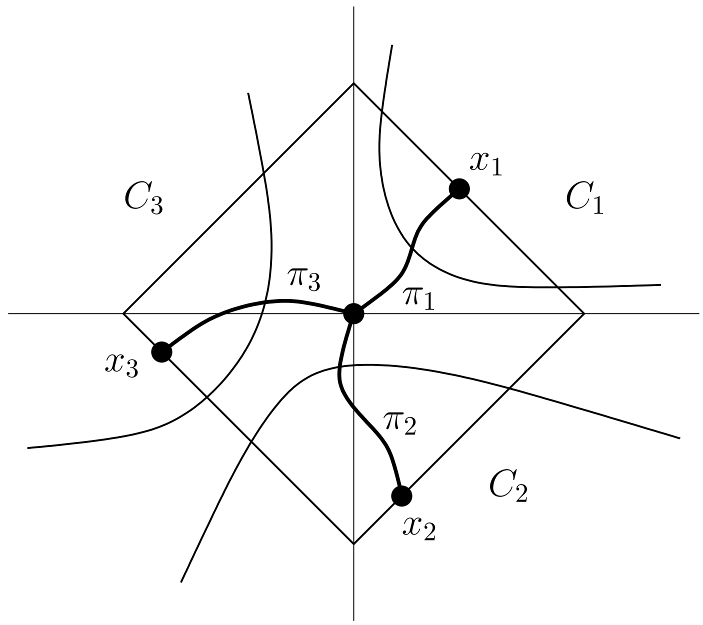

Figure 2: Making the origin a trifurcation.

Note that the infinite cluster containing x_i may not be C_i itself since C_i includes some edges from $E_{S(n^*)}$ which are now closed. And not every vertex in $\partial S(n^*) \cap C_i$ is a candidate for x_i, but there must be one such candidate which will be part of an infinite cluster contained in C_i.

Having found these three vertices, we argue that it is possible to find three edge disjoint paths from the origin to them (see Figure 2). This follows from geometry and we will not attempt to prove it rigorously. Let these three edge disjoint paths be π_1, π_2, and π_3 connecting the origin to x_1, x_2, and x_3 respectively.

Define the event E as follows: The edges of π_1, π_2, π_3 are open, and all other edges of $E_{S(n^*)}$ are closed. Having defined this we now see that if E occurs then the origin is a trifurcation. So, we can say that

$$E \cap \{M_{S(n^*)}(0) \geq 3\} \Rightarrow T_0.$$

The event E depends on the edges of $E_{S(n^*)}$ and the event $\{M_{S(n^*)}(0) \geq 3\}$ only depends on edges outside $E_{S(n^*)}$. So we have that

$$\mathrm{P}_p(T_0) \quad \geq \quad \mathrm{P}_p(E)\mathrm{P}_p(M_{S(n^*)}(0) \geq 3)$$

$$\geq \frac{1}{2}(min(p, 1-p))^{|E_{S(n^*)}|}.$$

Denoting the right hand side of the second inequality by γ, we can revisit (1) and rewrite it as follows

$$\mathrm{E}_p\left(\sum_{x\in B(n)} I_{T_x}\right) \geq \gamma|B(n)|. \tag{3}$$

So we have shown that the number of trifurcations grows linearly with $|B(n)|$ under the assumption that $\mathrm{P}_p(N=\infty)=1$. We will now show that the number of trifurcations inside $B(n)$ is upperbounded by $|\partial B(n)|$. This will lead to a contradiction since $|\partial B(n)| = o(|B(n)|)$.

In order to upperbound the number of trifurcations we will show that two distinct trifurcations in the same cluster can lay claim to distinct portions of $\partial B(n)$. In order to formalize this we study combinatorial structures called compatible 3-partitions.

Given a finite set Y, a *3-partition* of Y, $\pi = \{P_1, P_2, P_3\}$ is a partition of Y into 3 non-empty sets. Two 3-partitions are said to be *compatible* if there is a renumbering of π and π' such that

$$P_1 \supseteq P_2' \cup P_3'$$

and

$$P_1' \supseteq P_2 \cup P_3.$$

Note that these two conditions are actually the same, since $P_1 \supseteq P_2' \cup P_3'$ implies that $P \setminus P_1 \subseteq P \setminus (P_2' \cup P_3')$. Which is the same as saying that $P_1' \supseteq P_2 \cup P_3$. A set of 3-partitions $\mathbb{P}$ is called a *compatible family* if each pair $P_1, P_2 \in \mathbb{P}$ is compatible. The size of any compatible family is closely related to the size of the set being partitioned by the following lemma.

Lemma 9.5 *If $\mathbb{P}$ is a compatible family of distinct 3-partitions of Y,*

$$|\mathbb{P}| \leq |Y| - 2.$$

Proof of Lemma 9.5. The proof proceeds by induction. Since the smallest possible set to have a 3-partition has 3 elements and only one 3-partition, the base case holds.

Let us assume the result holds for all compatible families of 3-partitions of sets of size n and smaller. now consider Y such that $|Y| = n + 1$. Let us say that $\mathbb{P}$ is a compatible family for Y. Now, for some $y \in Y$, define $Z = Y \setminus \{y\}$. Each $\pi \in \mathbb{P}$ can be written as $\pi = (\{y\} \cup P_1, P_2, P_3)$. Based on this we partition $\mathbb{P}$ into two sets

1. $\mathbb{P}'$ where $P_1 \neq \emptyset$.
2. $\mathbb{P}''$ where $P_1 = \emptyset$.

Clearly $\mathbb{P}'$ with y removed from each P_1 is a compatible family for Z. So, by the induction hypothesis

$$|\mathbb{P}'| \leq |Z| - 2 = |Y| - 3.$$

Now, we show that $\mathbb{P}''$ has at most one element. Suppose $\mathbb{P}''$ contains 2 distinct elements, $(\{y\}, P_1, P_2)$ and $(\{y\}, P_1', P_2')$. Let the compatible renumbering be $(P_1, P_2, \{y\})$ and $(P_1', P_2', \{y\})$. This means that $P_2 \subseteq P_1' \cup \{y\}$. But this is not possible since $y \notin P_2$. Nor can any compatible renumbering have $P_1 = \{y\}$. Hence $|\mathbb{P}''| \leq 1$ and the result follows. ■

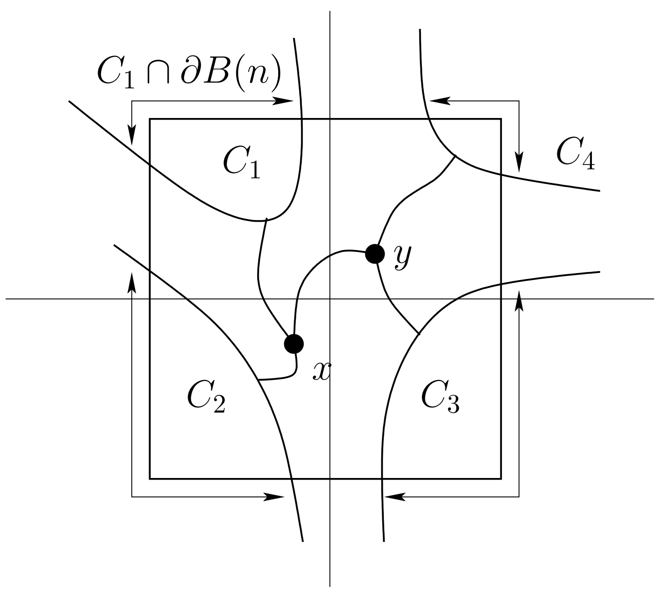

Figure 3: Two trifurcations of one open cluster K in $B(n-1)$.

Now let us relate compatible families to trifurcations. Let K be a cluster of $B(n)$. If x is a trifurcation of $K \cap B(n-1)$, removal of x induces a three partition on $K \cup \partial B(n)$. So, each $x \in K \cap B(n-1)$ corresponding to a 3-partition $\pi_x = \{P_1, P_2, P_3\}$ of $K \cap \partial B(n)$ such that

- P_i is non-empty for $i = 1, 2, 3$.
- P_i is a subset of a connected open subgraph of $B_n \setminus \{x\}$ for $i = 1, 2, 3$.
- $P_i \not\leftrightarrow P_j$ in $B(n) \setminus \{x\}, i \neq j$.

In Figure 3, for example, the 3-partition induced by x on $K \cap \partial B(n)$ is $\{(C_1 \cup C_2) \cap \partial B(n), C_2 \cap \partial B(n), C_3 \cap \partial B(n)\}$. Also in Figure 3 we can see that the two partitions induced by vertices x and y lying in the same open cluster K of $B(n-1)$ are compatible.

Hence, the number of trifurcations in K, denoted $\tau(K)$ must be upper-bounded by the size of the largest possible compatible family of 3-partitions of $K \cap \partial B(n)$ i.e., by Lemma 9.5

$$\tau(K) \leq |K \cap \delta B(n)| - 2.$$

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Comparing this to (3) we conclude that there is an n_0 (depending only on the value of γ) such that for all $n > n_0$ we have a contradiction. ∎