Lecture 8: Emulating the faulty mesh up to the critical probability

29th October 2008

8.1 Introduction

In the last lecture, we introduced several useful tools in the build up for an emulation of an $n \times n$ fault-free mesh by an $n \times n$ $(1-p)$-faulty mesh for any $p > p_c$ with $\mathcal{O}(\log n)$ slowdown. In this lecture, give the details of the emulation. This extends the result of [2], described in Lecture 6, which held only for "small" constant p, and makes it the best possible since for $p \leq p_c$ a large component does not exist with high probability and so any embedding would have a $\omega(1)$ load.

8.2 Notation

The notation used here will be introduced as needed. For a complete list of notation used, see Lecture 7. A few general remarks follow. In the following, the terms "cluster" and "component" are used interchangeably. p is the probability of an edge being open, i.e. non-faulty. $C(x)$ is the open component containing $x \in \mathbb{Z}^2$. $\mathbf{0}$ denotes the origin of the lattice, i.e. the point $(0, 0)$. C is shorthand for $C(\mathbf{0})$. $\chi(p)$ is the expected size of the component containing the origin, i.e. $\chi(p) = \mathrm{E}_p(|C|)$.

8.3 Supporting theorems

We will first prove two useful theorems concerning open paths from left to right in $T(n)$.

8.3.1 Existence of path connecting left to right face of $T(n)$

Theorem 8.1 *For $p > p_c$, given a box $T(n)$, the probability that there is an open path joining the left face of $T(n)$ to the right face is at least $1 - ne^{-\sigma n}$ for some $\sigma = \sigma(p) > 0$.*

Proof: Let us define two events A_n and B_n for the lattice $\mathbb{L}^2$ and it's dual lattice $\mathbb{L}^2_d$. $T^d(n)$ is the corresponding box in the dual lattice.

- $A_n :=$ There is an open path from left to right face of $T(n)$
- $B_n :=$ There is a closed path from top to bottom face of $T^d(n)$

Note that A_n and B_n are mutually exclusive and collectively exhaustive, which can be easily seen in figure 1: If A_n does not happen, it means that there should be a cut consisting entirely of closed edges which separates the left face from the right face, and thus B_n must happen. Conversely, if A_n happens, each cut from the left to the right face must contain at least one open edge and thus B_n cannot happen. This is captured by the following equation:

$$\mathrm{P}_{1-p}(B_n) + \mathrm{P}_p(A_n) = 1 \tag{1}$$

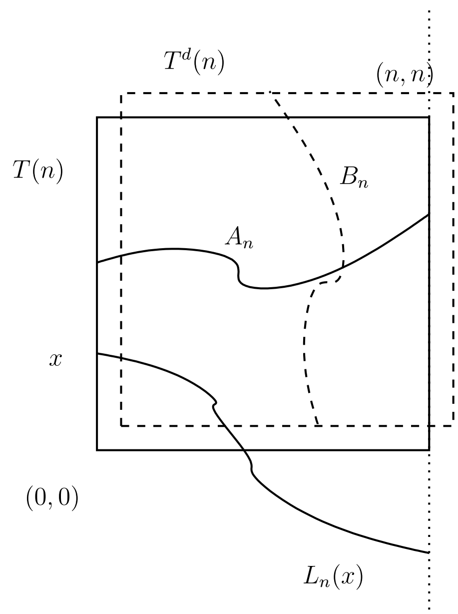

Figure 1: Events A_n, B_n and $L_n(x)$

We want to lower bound the probability $\mathrm{P}_p(A_n)$, so we can also give an upper bound to $\mathrm{P}_{1-p}(B_n)$. To do this we define the events $L(i)$ for $i \in S = \{0, 1, \ldots, n\}$, which occur when there is a line from $(0, i)$ which crosses the vertical line passing through $(n, 0)$. $L(S)$ occurs if any one of $L(i)$ occurs (see figure 1).

A_n implies the existence of a line from the left face of $T(n)$ to the line passing through $(n, 0)$, so $\mathrm{P}_p(A_n) \leq \mathrm{P}_p(L_n(S)$. Because the line is infinite, the events are translation-invariant and their probability is the same:

$$\mathrm{P}_p(A_n) \leq \mathrm{P}_p(L_n(S)) \tag{2}$$
$$\mathrm{P}_p(A_n) \leq n \cdot \mathrm{P}_p(L_n(0)) \tag{3}$$

If there is an open path from the origin $\mathbf{0}$ crossing the vertical line passing through $(n, 0)$, it also has to cross the border of the box $\vartheta B(n)$ somewhere:

$$\mathrm{P}_p(A_n) \leq n \cdot \mathrm{P}_p(\mathbf{0} \leftrightarrow \vartheta B(n)) \tag{4}$$

The same argument can be made for the dual symmetrically to get the upper bound on $\mathrm{P}_{1-p}(B_n)$:

$$\mathrm{P}_{1-p}(B_n) \leq n \cdot \mathrm{P}_{1-p}(\mathbf{0}_d \leftrightarrow \vartheta B(n)) \tag{5}$$
$$\leq n \cdot e^{-\sigma n} \qquad \text{by Theorem 7.6} \tag{6}$$

Therefore

$$\mathrm{P}_p(A_n) \geq 1 - n \cdot e^{-\sigma n} \qquad \text{combine (6) and (1)} \tag{7}$$

■

However, for the emulation we need multiple paths from the left face to the right face of $T(n)$. This is shown in the next section.

8.3.2 Existence of multiple paths from left to right face of $T(n)$

Theorem 8.2 *Given $p > p_c$ and a box $T(n)$ there exists a constant α so that the probability that there are at least $\alpha \cdot n$ open paths joining the left face $T(n)$ to the right face is at least $1 - e^{-\gamma n}$ for $\gamma = \gamma(p) > 0$.*

Proof: In the proof we need the notion of a sphere around an event. Let Ω be the set of all outcomes of the bond percolation in $\mathbb{L}^2$. If $\omega \in \Omega$ then we define $S_r(\omega)$ as the sphere of radius r around ω in Ω such that

$$S_r(\omega) = \{\omega' \in \Omega : \sum_{e \in \mathbb{E}^2} |\omega(e) - \omega'(e)| \leq r\}$$

$S_r(\omega)$ contains all outcomes which differ from ω in at most r edges. Given an event $A \subseteq \Omega$, define $I_r(A)$, the r-interior of the event A, as

$$I_r(A) = \{\omega \in \Omega : S_r(\omega) \subseteq A\}$$

In other words, $I_r(A)$ contains all outcomes in which up to r edges can be changed and which still are in A after that.

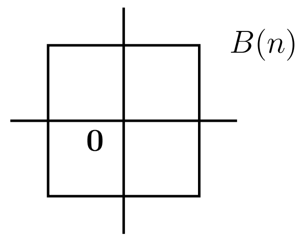

Figure 2: $B(n)$ with all edges open

An example may help to understand the definitions: Event A occurs if the origin $\mathbf{0}$ is connected to the border of $B(n)$, i.e. $A := \mathbf{0} \leftrightarrow \vartheta B(n)$. In the grid displayed in Figure 2, $I_r(A)$ is empty for $r \geq 4$, because if the 4 edges around the origin are closed, A cannot occur. For $r = 3$, $I_r(A)$ contains at least the outcome which has all the edges in $B(n)$ open.

We recall Menger's Theorem to establish the connection between the $I_r(A_n)$ and the existence of edge-disjoint paths.

Theorem 8.3 (Menger's theorem) *If $A, B \subseteq V(G)$ have the property that $A \leftrightarrow B$ even if up to $k-1$ edges of G are removed, then there are k edge disjoint paths in G from A to B.*

Let us look at the event $I_r(A_n)$, where A_n occurs, if there is a path from the left edge to the right edge in $B(n)$. All outcomes in $I_r(A_n)$ are resistant to r edge removals, so according to Theorem 8.3, there are $r + 1$ edge disjoint paths from the left side to the right side. Now we only have to show that the probability of $I_r(A_n)$ for $r = \alpha n$, where $\alpha > 0$ is some small enough constant, approaches 1 as $n \to \infty$.

To do this we use following Theorem 8.4 (proof omitted) to show that the probability of the complementary event approaches 0 as $n \to \infty$. For the definition of increasing event, see Lecture 7.

Theorem 8.4 *[1] For an increasing event A and an integer r*

$$1 - P_{p_2}(I_r(A)) \le \left(\frac{p_2}{p_2 - p_1}\right)^r (1 - P_{p_1}(A))$$

whenever $0 \le p_1 < p_2 \le 1$

In the above theorem set $A = A_n$ and $p_c < p_1 < p_2 = p$. Then

$$\begin{aligned}
1 - \mathrm{P}_{p_2}(I_r(A_n)) &\le \left(\frac{p_2}{p_2 - p_1}\right)^r (1 - \mathrm{P}_{p_1}(A_n)) \\
&= \left(\frac{p_2}{p_2 - p_1}\right)^r \cdot \mathrm{P}_{p_1}(B_n) && \text{by (1)} \\
&\le \left(\frac{p_2}{p_2 - p_1}\right)^r \cdot ne^{-\sigma(p_1)n} && \text{by Theorem 8.1} \\
&\le e^{\log\left(\frac{p_2}{p_2 - p_1}\right)^r + \log n + \log e^{-\sigma(p_1)n}} \\
&= ne^{r \log \frac{p_2}{p_2 - p_1} - \sigma(p_1)n}
\end{aligned}$$

If we set $r = \alpha n$ we get

$$= ne^{(-\sigma(p_1) + \alpha \log \frac{p_2}{p_2 - p_1})n}$$

By choosing α appropriately, the exponent can be made negative. The leading factor n in (8) can be ignored because it will be dominated by the rapidly decaying exponential, so:

$$1 - \mathrm{P}_p(I_{\alpha n}(A_n)) \le \underbrace{ne^{-\lambda(p)n}}_{\to 0 \text{ for } n \to \infty} \tag{8}$$

with $\lambda(p) > 0$. ■

8.4 Inapplicability of the "highway" construction

In the previous section we showed that there exist αn paths joining the left and right face of $T(n)$ w.h.p. But is this enough? We recall the "highway" construction of Kaklamanis et. al.

In this construction we tiled the $n \times n$ grid with $r \times r$ grid-like blocks. If each grid had at least $\frac{2}{3}r$ paths from left to right, then we can construct at least $\frac{1}{3}r$ paths from the left side to the right side of $T(n)$. For full details of the construction, see Lecture 6 or [2].

But Theorem 8.2 does not guarantee $\alpha \ge \frac{2}{3}$. Thus we cannot use the construction described above directly. We instead use a different tiling strategy.

8.5 A "domino" tiling strategy

Define the box $T(l, kl)$ as the box with lower left corner at $(0, 0)$, height l and width kl. We define the following event for some $\alpha > 0$:

$$LR(l, kl) : \text{ The left and right face of } T(l, kl) \text{ are connected}$$

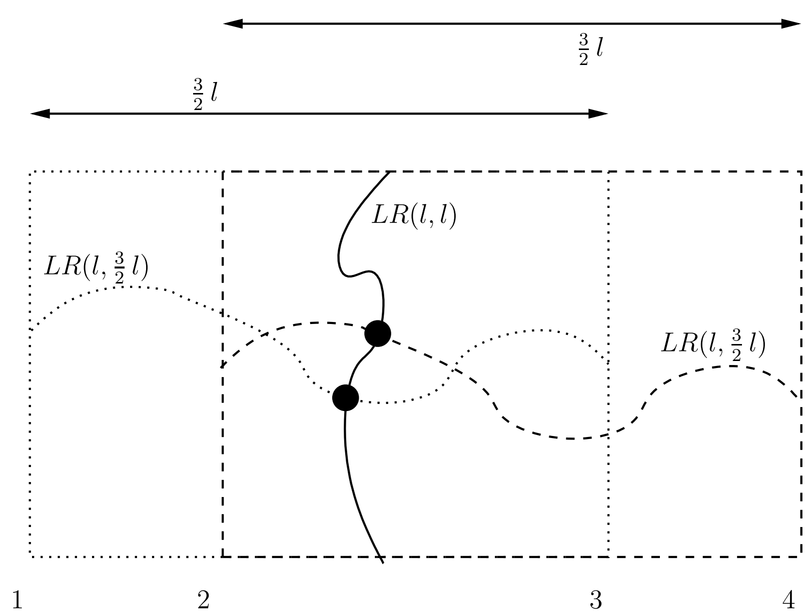

Figure 3: Events $LR(l_1, l_2)$

The dominoes are of size $T(l, \frac{3}{2}l)$ and are nested as displayed in Figure 3. Then we have the following relation:

Lemma 8.1 *[4]*

$$P_p(LR(l, 2l)) \geq P_p(LR(l, l)) \cdot P_p(LR(l, \tfrac{3}{2}l))^2$$

Proof: See Figure 3 for the basic idea of the proof. If we have a path from face 1 to face 3 and a path from face 2 to face 4, we can use the vertical path in the common area to connect up the paths. Since all the three events are increasing events, we can use the FKG inequality, which yields the result immediately. ■

We also have the following bound on $\mathrm{P}_p(LR(l, \frac{3}{2}\,l))$ in terms of $\mathrm{P}_p(LR(l,l))$ (which we know). The proof is omitted.

Theorem 8.5 *[4], [5] For $\tau = P_p(LR(l,l))$*

$$P_p(LR(l, \tfrac{3}{2}l)) \geq (1 - \sqrt{1-\tau})^3$$

.

In the above theorem, set $\tau = (1 - e^{-\gamma(p)l})$ and use Lemma 8.1

$$\begin{aligned}
\mathrm{P}_p(LR(l,2l)) &\geq \tau \cdot \mathrm{P}_p(LR(l, \tfrac{3}{2}l))^2 && \tau \text{ in } (8.1)\\
&\geq (1 - e^{-\beta(p)l}) \cdot \mathrm{P}_p(LR(l, \tfrac{3}{2}l))^2 && \text{by Theorem 8.2}\\
&\geq (1 - e^{-\beta(p)l}) \cdot (1 - e^{\frac{-\beta(p)}{2}l})^{2\cdot 3} && \text{by Theorem 8.5}\\
&\geq (1 - e^{-\beta(p)l})^7
\end{aligned}$$

if we leave out the positive terms in the sum, we get

$$\mathrm{P}_p(LR(l,2l)) \geq 1 - 7e^{-\beta(p)l}$$

for large l you can find a constant $\psi(p) > 0$, so that

$$\mathrm{P}_p(LR(l,2l)) \geq 1 - e^{-\psi(p)l}$$

With the same argument as in the proof for Theorem 8.2 we can argue that there must exist some α and an appropriate $\eta(p)$, so that:

$$\mathrm{P}_p(I_{\alpha l}(LR(l,2l))) \geq 1 - e^{-\eta(p)l} \tag{9}$$

In the above, set $l = \frac{3\log n}{\eta(p)}$ (the size of the "dominoes") to get the result that with probability at least $1 - \mathcal{O}(\frac{1}{n^3})$ we have a "good domino", i.e. a domino which has open paths from the left face to the right face of the domino, which can be tiled together to yield open paths from the left face to the right face of $T(n)$. Since there are only $\mathcal{O}\left(\left(\frac{n}{\log n}\right)^2\right)$ dominoes, the probability that they are all "good" is at least $1 - \mathcal{O}(\frac{1}{n})$. Thus we have shown the following theorem:

Theorem 8.6 *[3] For any $p > p_c$ it is possible to emulate a $n \times n$ mesh by an $n \times n$ $(1-p)$ - faulty mesh with $\mathcal{O}(l + C + D)$ slowdown where $l = \mathcal{O}(1)$, $C = \mathcal{O}(1)$ and $D = \mathcal{O}(\log n)$.*

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References

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