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Lecture 7: The critical probability for bond percolation in 2 dimensions is $\frac{1}{2}$

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7.1 Introduction

From percolation theory we know that there exists a probability p_c such that $\forall p > p_c$, there exists an infinite component in an infinite mesh and a $\theta(n^2)$ component in a $n \times n$ mesh with high probability. We saw the Kaklamanis result [2] in the previous lecture, which states that $\exists p'$: for all $p > p'$, a $n \times n$ mesh can emulate a fault-free $n\sqrt{\log_{\frac{1}{p}} n} \times n\sqrt{\log_{\frac{1}{p}} n}$ mesh with $\mathcal{O}(\log_{\frac{1}{p}} n)$ slowdown, where $p' \geq p_c$. We will prove in the following lectures that it is possible to get a similar emulation with $\mathcal{O}(\log n)$ slowdown, for all $p > p_c$. Note p in the last lecture was the fault probability, here it refers to the probability of existence of edge.

In this lecture we will prove that critical probability p_c for infinite mesh is 1/2. This result will be used in subsequent lectures for proving that an emulation with $\mathcal{O}(\log n)$ slowdown can be done for all $p > p_c$.

7.2 Notations and Definitions

In percolation theory, each edge in a graph is open with probability p, or closed with probability $1 - p$. Each edge is open or closed independent of other edges.

7.2.1 2-D Lattice

We represent a 2-dimensional lattice on integer point as $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$. $\mathbb{Z}^2$ denotes the set of 2-dimensional integer points. $\mathbb{E}^2$ denotes the set of edges and is defined as follows:

$$\mathbb{E}^2 = \{(x, y) : x, y \in \mathbb{Z}^2, \sum_{i=1}^{2} |x_i - y_i| = 1\}$$

The percolated lattice is formed when every edge is this lattice exists with probability p. Now for every edge $e \in \mathbb{E}^2$ we define a random variable X_e as follows,

$$\mathrm{P}_e\left(X_e = 1\right) = p \qquad\qquad (e \text{ is open})$$

$$\mathrm{P}_e\left(X_e = 0\right) = 1 - p \qquad\qquad (e \text{ is closed})$$

Note that the random variables X_e are independent. i.e the probability of an edge e_1 being open, is independent of the probability of another edge e_2 being open.

Now, we define the product measure of $\mathbb{L}^2$ as follows,

$$P = \prod_{e \in \mathbb{E}^2} \mathrm{P}_e$$

We also define an outcome vector ω in which the i^{th} entry, $\omega(i)$, is set 1 if the edge e_i is open and 0 if it is closed. Since, $\mathbb{L}^2$ is an infinite mesh, there are an infinite no. of edges and, therefore, there are infinite entries in ω. Hence, the probability of any outcome is a 0 measure.

Lets define Ω to be the set of all outcome vectors.

We can define a partial order (a transitive, antisymmetric and reflexive relation) on Ω as follows,

For ω_1, $\omega_2 \in \Omega$, we say that $\omega_1 \leq \omega_2$, if

$$\forall e \in \mathbb{E}^2,\, \omega_1(e) \leq \omega_2(e)$$

i.e. Every edge that is open in ω_1, is also open in ω_2.

7.2.2 General Notation

1. $\mathrm{P}_p(A)$: Probability of an event A occurring, where each edge is open (edge exists) with probability p.
2. ∂S: Denotes boundary of set S. We can see that $\partial S \subseteq S$.
3. $C(x)$: Denotes the open component containing $x \in \mathbb{Z}^2$. By an open component we mean a connected component formed from open edges, and for convenience $C(\mathbf{0})$ is denoted as C, where $\mathbf{0}$ is the origin.

4. $x \leftrightarrow y$: Denotes the event that **x** is connected to **y** through a sequence of open edges.

5. $\{A\}$: Denotes set of outcomes in which event A happens.

6. $\chi(p)$: It is defined as the expected size of cluster (or component) containing the origin **0** when each edge is open with probability p, $\chi(p) = \mathrm{E}_p(|C|)$

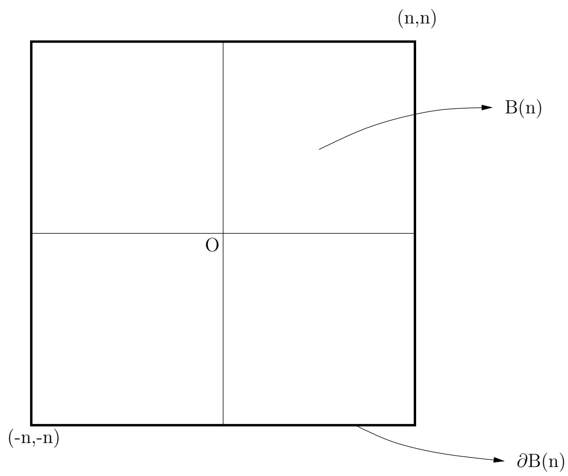

Figure 1: Bounding box $B(n)$ with boundary $\partial B(n)$

7.2.3 Increasing/Decreasing events

Definition A random variable $X : \Omega \to \mathbb{R}$ is called *increasing* (or *decreasing*) if $X(\omega_1) \leq X(\omega_2)$, whenever $\omega_1 \leq \omega_2$ (or $\omega_1 \geq \omega_2$).

Define $B(n)$ as the bounding box as shown in Figure 1.

Example Increasing random variable

- X : No. of open edges in $B(n)$.

Example Decreasing random variable

- X : No. of closed edges in $B(n)$.

An indicator variable for an event A is defined as follows,

$$I_A = \begin{cases} 1 \text{ if } A \text{ happens} \\ 0 \text{ if } A \text{ doesn't happen} \end{cases}$$

Definition An event A is called *increasing* (or *decreasing*) if its indicator variable I_A is increasing (or decreasing).

Example Consider the event B : There are even no. of open edges in $B(n)$.

We can't call B either an increasing or a decreasing event.

7.2.4 $A \circ B$

Definition Given increasing events A and B, $A \circ B$ is the event that A and B "happen on disjoint sets of edges."

Formally this is stated as follows,
Given $\omega \in \Omega$, $K(\omega) = \{e : \omega(e) = 1\}$, we say that $\omega \in A \circ B$, if $\exists\ \omega_1, \omega_2$ s.t.

- $\omega_1 \in A$, $\omega_2 \in B$
- $K(\omega_1) \cap K(\omega_2) = \emptyset$
- $K(\omega_1) \cup K(\omega_2) \subseteq K(\omega)$

Note that $A \circ B \subseteq A \cap B$. This follows from the observation that $A \cap B$ also includes those outcomes in which both A and B happen on overlapping edges.

$$\therefore \mathrm{P}(A \circ B) \leq \mathrm{P}(A \cap B)$$

7.3 Basic tools from probability theory

Theorem 7.1 *For an increasing event A and* $p_1 \leq p_2 \leq 1$

$$P_{p_1}(A) \leq P_{p_2}(A)$$

Proof. Suppose two different experiments are being performed. In Experiment 1 (Ex_1) each edge is being retained with probability p_1 and with probability p_2 in Experiment 2 (Ex_2). We proceed to show that whenever event A happens in Ex_1 it also happens in Ex_2. We prove this by relating the two experiments using a technique called coupling which is done as follows: $\forall e \in \mathbb{E}^2$ we associate a random variable $Y_e \in$ [0,1]. Then we proceed as follows,

- Declare e open in Ex_1, if $Y_e < p_1$
- Declare e open in Ex_2, if $Y_e < p_2$

And since $p_1 \leq p_2$, whenever an edge e is open in Ex_1 then it is open in Ex_2 also. Since A is an increasing event if it happens in Ex_1, it will also happen in Ex_2. Hence the theorem is proved.

Note that $\mathrm{P}(Y_e < p_1) = p_1$ and $\mathrm{P}(Y_e < p_2) = p_2$. So we are still retaining edges in the two experiments with their respective probabilities. ∎

Theorem 7.2 FKG inequality*: Given increasing events A and B,*

$$P(A \cap B) \geq P(A)P(B)$$

We must note that if A and B were both decreasing events (A and B are still positively correlated), even then the above inequality would hold.

Similarly, if A and B are negatively correlated then the inequality would get reversed.

Example Consider $B(n)$ as shown in Figure 1.

Let A be the event that an open path exists from left to right in $B(n)$ and B the event that an open path exists from top to down in $B(n)$.

Both A and B are increasing events. By intuition, we can see that if B happens, i.e. a path exists from top to down then the probability that A happens can only improve. Hence, we see that the FKG Inequality holds, i.e. A and B are positively correlated.

Theorem 7.3 BK inequality: *Given increasing events A and B*

$$P(A \circ B) \leq P(A)P(B)$$

For further details regarding the above inequalities, one can refer [1].

Lemma 7.1 ***(Square root trick)*** *Given a set of increasing events $A_1, \ldots, A_m$ of equal probability.*

$$P_p(A_1) \geq 1 - \left\{1 - P_p\left(\bigcup_{i=1}^{m} A_i\right)\right\}^{\frac{1}{m}}$$

Proof. Probability that none of the events A_i occurs can be given as,

$$\begin{aligned} 1 - \mathrm{P}_p\left(\bigcup_{i=1}^{m} A_i\right) &= \mathrm{P}_p\left(\bigcap_{i=1}^{m} \bar{A}_i\right) \\ &\geq \prod_{i=1}^{m} \mathrm{P}_p\left(\bar{A}_i\right) \text{ (Using FKG Inequality)} \\ &= \left(1 - \mathrm{P}_p\left(A_1\right)\right)^m \text{ (Since } A_i\text{'s are of equal probability)} \end{aligned}$$

$$\therefore 1 - \mathrm{P}_p\left(\bigcup_{i=1}^{m} A_i\right) \geq \left(1 - \mathrm{P}_p\left(A_1\right)\right)^m$$

$$\Rightarrow \left\{1 - \mathrm{P}_p\left(\bigcup_{i=1}^{m} A_i\right)\right\}^{\frac{1}{m}} \geq 1 - \mathrm{P}_p\left(A_1\right)$$

$$\Rightarrow 1 - \left\{1 - \mathrm{P}_p\left(\bigcup_{i=1}^{m} A_i\right)\right\}^{\frac{1}{m}} \leq \mathrm{P}_p\left(A_1\right)$$

Hence, proved. ■

7.4 Critical probability for percolation in 2-dimensions

Now, let us define the critical probability of a mesh p_c in terms of the above notations.

If $p \leq p_c$, $\mathrm{P}_p(|C| > k) \rightarrow 0$ as $k \rightarrow \infty$

i.e. If the probability of an edge being open p is less than or equal to the critical probability p_c, then the probability of finding an infinite sized open cluster, containing the origin, is 0.

Theorem 7.4 *If $p \leq p_c$, then $\chi(p) < \infty$*

Theorem 7.5 *If $\chi(p) < \infty$, there exists $\sigma(p) > 0$ s.t.*

$$P_p\left(\mathbf{0} \leftrightarrow \partial B(n)\right) \leq e^{-n\sigma(p)}$$

Note that $\sigma(p)$ is independent of n. The term on the left of the inequality represents the probability that an open path exists from the origin to $\partial B(n)$. This probability must increase as p increases. Therefore, we expect that $\sigma(p)$ should decrease as p increases, which is indeed the case.

Proof. Define a random variable N_n as the no. of nodes of $\partial B(n)$ to which the origin is connected through open paths.

For $x \in \partial B(n)$, $\tau_p(\mathbf{0}, x) = \mathrm{P}_p\left(\mathbf{0} \leftrightarrow x\right)$

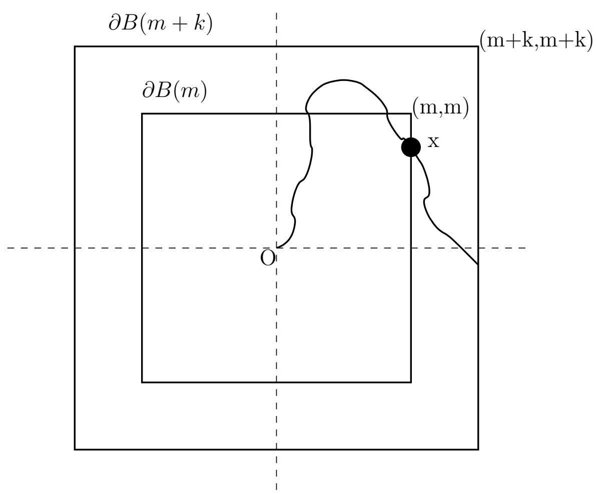

Figure 2: A path from origin to $\partial B(m+k)$

We can see that,

$$\{\mathbf{0} \leftrightarrow \partial B(m+k)\} \subseteq \bigcup_{x \in \partial B(m)} (\mathbf{0} \leftrightarrow x) \circ (x \leftrightarrow \partial B(k, x))$$

Consider any path from the origin to the boundary of $B(m+k)$, see Figure 2. It must have intersected the boundary of $B(m)$ at some node. Lets choose that x which corresponds to the last node which is intersected on $\partial B(m)$. Lets define $\partial B(k, x)$ as the boundary of the box of side length

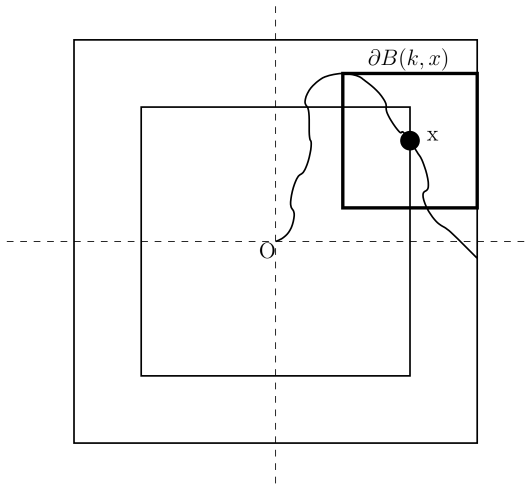

Figure 3: B(k,x), Box of size k around x

$2k$ centered at node x, see Figure 3. Also, since there is a path from this x to $\partial B(m+k)$, there must be a path from x to $\partial B(k,x)$. Since, we chose x to be the last node to be intersected on $\partial B(m)$, $\mathbf{0} \leftrightarrow x$ & $x \leftrightarrow \partial B(k,x)$ occur on a disjoint set of edges. But not all paths on the RHS correspond to a path that reaches $\partial B(m+k)$. Hence, the LHS is a subset of the RHS.

From this we get,

$$\begin{aligned} \mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(m+k)\right) &\leq \sum_{x \in \partial B(m)} \mathrm{P}_p\left((\mathbf{0} \leftrightarrow x) \circ (x \leftrightarrow \partial B(k,x)\right) \\ &\leq \sum_{x \in \partial B(m)} \mathrm{P}_p\left(\mathbf{0} \leftrightarrow x\right).\mathrm{P}_p\left(x \leftrightarrow \partial B(k,x)\right) \text{ (Using BK inequality)} \\ &= \sum_{x \in \partial B(m)} \tau_p(\mathbf{0}, x).\mathrm{P}_p\left(x \leftrightarrow \partial B(k,x)\right) \end{aligned}$$

From the translational invariance property $\forall x \in \mathbb{Z}^2$, $\mathrm{P}_p\left(x \leftrightarrow \partial B(k,x)\right) = \mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(k)\right)$. Using this we get,

$$\mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(m+k)\right) \leq \sum_{x \in \partial B(m)} \tau_p(\mathbf{0}, x).\mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(k)\right)$$

$$\therefore \mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(m+k)\right) \leq \mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(k)\right).\left(\sum_{x \in \partial B(m)} \tau_p(\mathbf{0}, x)\right) \tag{1}$$

Now, lets define the no. of nodes on the boundary of $\partial B(m)$ which have an open path which connects them to the origin by N_m.

i.e. $N_m = \sum_{x \in \partial B(m)} I_{\mathbf{0} \leftrightarrow x}$, where

$$I_{\mathbf{0} \leftrightarrow x} = \begin{cases} 1 \text{ if } \mathbf{0} \leftrightarrow x \\ 0 \text{ otherwise} \end{cases}$$

$$\therefore \mathrm{E}_p\left(N_m\right) = \sum_{x \in \partial B(m)} \tau_p(\mathbf{0}, x) \tag{2}$$

Substituting (2) into (1) gives us,

$$\mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(m+k)\right) \leq \mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(k)\right).\mathrm{E}_p\left(N_m\right) \tag{3}$$

Let's say we get an m^* s.t. $\mathrm{E}_p\left(N_m\right) < 1$. We will show that the theorem holds if we do find such an m^*. Then we will show that such an m^* actually does exist.

Suppose, $n = s.m^* + r$, where $s \geq 0$ & $0 \leq r < m^*$. Then using (3) we get,

$$\begin{aligned}
\mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(n)\right) &\leq \mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(n-m^*)\right).\mathrm{E}_p\left(N_{m^*}\right) \\
&\leq \mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(n-2m^*)\right).\left(\mathrm{E}_p\left(N_{m^*}\right)\right)^2 \\
&\vdots \\
&\leq \mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(r)\right).\left(\mathrm{E}_p\left(N_{m^*}\right)\right)^s \\
&\leq \left(\mathrm{E}_p\left(N_{m^*}\right)\right)^s \\
&\leq \left(\mathrm{E}_p\left(N_{m^*}\right)\right)^{\left(\frac{n}{m^*}-1\right)} \quad \left(\because \mathrm{E}_p\left(N_{m^*}\right) < 1\right) \\
&= e^{\log\left(\mathrm{E}_p\left(N_{m^*}\right)\right)\left(\frac{n}{m^*}-1\right)} \\
&\leq e^{\log\left(\mathrm{E}_p\left(N_{m^*}\right)\right)\left(\frac{n}{2m^*}\right)} \quad \left(\because \mathrm{E}_p\left(N_{m^*}\right) < 1\right) \\
&= e^{-\left[\frac{\log\left(\frac{1}{\mathrm{E}_p\left(N_{m^*}\right)}\right)}{2m^*}.n\right]}
\end{aligned}$$

Set $\sigma(p) = \frac{\log\left(\frac{1}{\mathrm{E}_p(N_{m^*})}\right)}{2m^*}$, which is independent of n and decreases with increase in p

$\therefore \mathrm{P}_p\left(\mathbf{0} \leftrightarrow \partial B(n)\right) \leq e^{-\sigma(p).n}$

But, now we need to find an m^* s.t. $\mathrm{E}_p\left(N_{m^*}\right) < 1$.

We can see that $|C| = \sum_{n=0}^{\infty} N_n$. This is easy to visualize as each node in C will lie on the boundary of some bounding box $B(n)$ and will be connected to the origin.

$\therefore \chi(p) = \mathrm{E}_p\left(|C|\right) = \sum_{n=0}^{\infty} \mathrm{E}_p\left(N_n\right)$

$\Rightarrow \sum_{n=0}^{\infty} \mathrm{E}_p\left(N_n\right) < \infty$ (Using Theorem 7.4)

$\Rightarrow \lim_{n\to\infty} \mathrm{E}_p\left(N_n\right) \to 0$

$\Rightarrow \exists m^*$ s.t. $\mathrm{E}_p\left(N_{m^*}\right) < 1$

This follows from the fact that if the infinite sequence tends to 0, at some point we will definitely find a value less than 1.

Hence, the theorem is proved. ■

Theorem 7.6 *If $p > p_c$, then there exists a unique infinite cluster.*

Refer [1] for proof.

7.5 The critical probability is $\frac{1}{2}$

Now we have the necessary tools to prove Harry Kesten's famous theorem on the critical probability for bond percolation in two dimensions.

Theorem 7.7 *[3] The critical probability p_c of bond percolation in a 2-dimensional lattice $\mathbb{L}^2$ is $\frac{1}{2}$.*

To show this we will first prove that $p_c \geq \frac{1}{2}$ and then that $p_c \leq \frac{1}{2}$.

7.5.1 First part: Show that $p_c \geq \frac{1}{2}$

Lemma 7.2 $p_c \geq \frac{1}{2}$

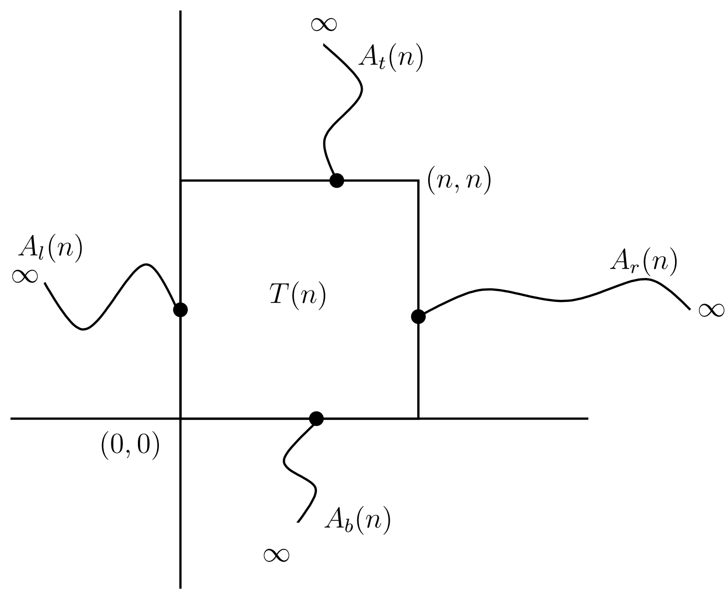

Figure 4: The events $A_r(n), A_t(n), A_l(n), A_b(n)$

Proof: Let $\theta(p)$ be the probability that the origin is part of an infinite component. We will show that $\theta(\frac{1}{2}) = 0$. Since $p_c = \sup\{p \,|\, \theta(p) = 0\}$, this is the same as saying that $p_c \geq \frac{1}{2}$.

Consider the box $T(n)$ with corners at $(0,0)$ and (n,n). Let $A_r(n)$ be the event that there exists an open path that starts at the right boundary of $T(n)$, doesn't go inside the box and goes to infinity. Define similar events $A_t(n), A_b(n), A_l(n)$ for the top, bottom and the left boundaries of $T(n)$ respectively (see Figure 4). Let $\mathcal{J}$ be the index set $\{l, r, t, b\}$.

Note that if $T(n)$ intersects the infinite cluster, then the event $\bigcup_{j \in \mathcal{J}} A_j(n)$ must occur. This means that for $p = \frac{1}{2}$

$$\mathrm{P}_p(T(n) \text{ intersects the } \infty\text{−cluster}) \leq \mathrm{P}_p(\bigcup_{j \in \mathcal{J}} A_j(n) \text{ happens})$$

We can check that as $n \to \infty$, the term on the left hand side $\to 1$. Also, note that the probability of all the four events is the same. Using the square root trick (Lemma 7.1), for any $j \in \mathcal{J}$:

$$\mathrm{P}_p(A_j) \geq \underbrace{1 - (\underbrace{1 - \underbrace{\mathrm{P}_p\left(\bigcup A_j\right)}_{\to 1 \text{ for } n\to\infty}}_{\to 0 \text{ for } n\to\infty})^{\frac{1}{4}}}_{\to 1 \text{ for } n\to\infty} \tag{4}$$

With increasing n we can get $\mathrm{P}_p(A_x)$ arbitrarily close to 1, so there exists an N' with $\mathrm{P}_{\frac{1}{2}}(A_x(N')) > \frac{7}{8}$.

We now work with the dual lattice. Define $A_j^d(n), j \in \mathcal{J}$ for the closed paths in the dual lattice similarly. Since $p = \frac{1}{2}$, $1 - p$ is the same as p and since the dual of the lattice is the lattice itself, the same argument can be used to show that there exists a number N'' such that $\mathrm{P}_p(A_j^d(N'')) = \mathrm{P}_{1-p}(A_j^d(N'')) > \frac{7}{8}$. We proceed with the bigger constant $N = \max(N', N'')$.

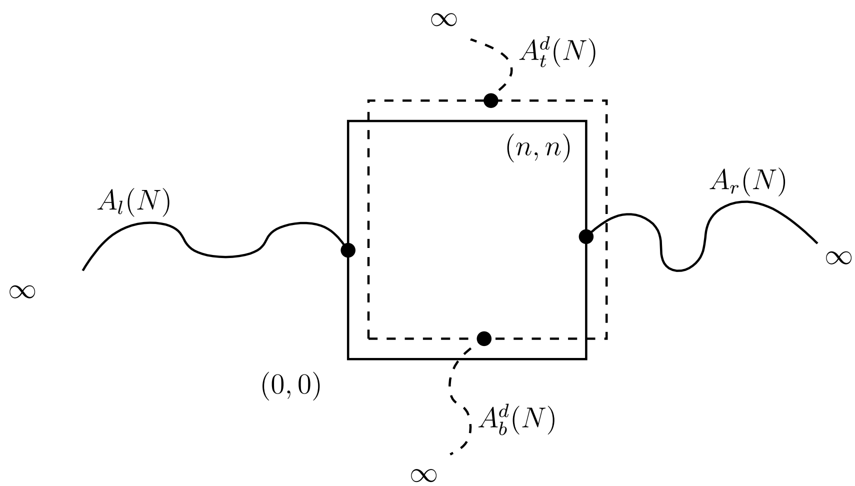

Figure 5: The event A

Now our goal is to define an event that would require the two components to intersect. Let $A = A_l(N) \cap A_r(N) \cap A_t^d(N) \cap A_b^d(N)$ be this event, which states that there is an infinite open path originating at the left and at the right edge in the primal and an infinite closed path originating at the top and at the bottom edge in the dual lattice (see Figure 5). To determine the probability of A we give an upper bound to the complementary event $\overline{A}$:

$$\begin{aligned}
\overline{A} &= \overline{A_l}(N) \cup \overline{A_r}(N) \cup \overline{A_t^d}(N) \cup \overline{A_b^d}(N) \\
\mathrm{P}(\overline{A}) &\leq \mathrm{P}(\overline{A_l}(N)) + \mathrm{P}(\overline{A_r}(N)) + \mathrm{P}(\overline{A_t^d}(N)) + \mathrm{P}(\overline{A_b^d}(N)) \\
\mathrm{P}(\overline{A}) &\leq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
\mathrm{P}(A) &\geq \frac{1}{2}
\end{aligned}$$

Because $\mathrm{P}(A)$ is positive, there is a possible outcome which leads to the contradiction. This can be seen as follows. Denote the paths corresponding to $A_j(n), j \in \mathcal{J}$ as $P_j(n)$. Define $P_j^d(n)$ similarly for the dual lattice. Since the ∞-component is unique (see Theorem 7.6), one of the following two cases must occur:

Case 1: $P_L(N)$ and $P_R(N)$ meet outside the box. Since these paths are always outside $T(N)$, they must intersect the paths corresponding to either $A_T^d(N)$ or $A_B^d(N)$. But this is not possible since an edge is either open or closed.

Case 2: There should be an open path inside $T(N)$ connecting $P_L(N)$ and $P_R(N)$. This means that there are two infinite components in the dual lattice (the paths corresponding to $A_T^d(N)$ and $A_B^d(N)$), which is again not possible.

By contradiction, we have shown that $\theta(\frac{1}{2}) = 0 \implies p_c \geq \frac{1}{2}$. ■

7.5.2 Second part: Show that $p_c \leq \frac{1}{2}$

To prove the other half of Theorem 7.7, we first need this lemma.

Lemma 7.3 *If $p < p_c$, then*

$$P_p(\textit{origin of } \mathbb{L}_d^2 \textit{ is part of an } \infty\textit{-component of closed edges}) > 0$$

Let us assume that Lemma 7.3 is true. We prove the following:

Lemma 7.4 $p_c \leq \frac{1}{2}$

Proof: Lemma 7.3 says that

$$p < p_c \qquad \Longrightarrow \theta(1-p) > 0 \tag{5}$$

$$\text{or } p < p_c \qquad \Longrightarrow 1-p > p_c \tag{6}$$

The above can only hold if $p_c \leq \frac{1}{2}$. To see this, suppose the converse holds, i.e. $p_c = \frac{1}{2} + \epsilon$. Then for $p = \frac{1}{2} + \epsilon/2$, Equation (6) is false. Thus we have shown by contradiction that $p_c \leq \frac{1}{2}$. ■

To complete the proof of Theorem 7.7, we now prove Lemma 7.3.

Proof: For a given $M > 0$ and $k \leq 0$, we define the event A_M as follows (see Figure 6). Event A_M occurs if $(k,0) \leftrightarrow (l,0)$ for some $l > M$ through an open path which lies wholly above the X-axis.

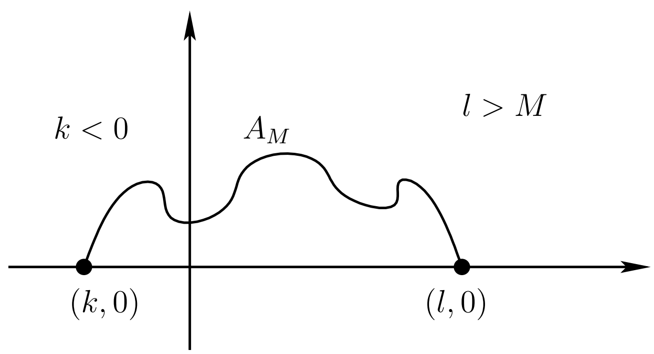

Figure 6: The event A_M

$$\begin{aligned} \mathrm{P}_p(A_M) &\leq \mathrm{P}_p\Big(\bigcup_{l=M}^{\infty}\{(l,0) \leftrightarrow (k,0)\}\Big) \\ &\leq \sum_{l=M}^{\infty} \mathrm{P}_p(|C(k,0)| \geq l) \\ &= \sum_{l=M}^{\infty} \mathrm{P}_p(|C| \geq l) \end{aligned}$$

Note that the last sum is the tail of the series:

$$\begin{aligned} & \sum_{l=1}^{\infty} \mathrm{P}_p(|C| \geq l) \\ = & \sum_{l=1}^{\infty} l \cdot \mathrm{P}_p(|C| = l) \\ = & \chi(p) \\ < & \infty \end{aligned}$$

The last inequality follows from the fact that $p < p_c \implies \chi(p) < \infty$ (see Theorem 7.4). The sum converges, and all terms are ≥ 0. So the terms must decrease, as l increases. This means we can leave out terms, i.e. find a M^* so that $\mathrm{P}_p(A_{M^*}) < \frac{1}{2} \implies \mathrm{P}_p(\overline{A_{M^*}}) > \frac{1}{2}$.

We now use an argument also used in lecture 1: A finite component of closed edges in the dual lattice must be surrounded by a circuit of open edges in the primal lattice. Consider the set $L = \{(m + \frac{1}{2}, \frac{1}{2}) \mid 0 \leq m < M^*\}$ in the dual lattice. Let $C^d(L)$ be the set of vertices connected to L through closed edges. If $C^d(L)$ is finite, then in the dual of the dual lattice (i.e. the primal lattice), there exists a closed cycle enclosing $C^d(L)$. Note that the upper part of the circuit connects the negative part of the X-axis to some $(l, 0)$ where $l > M^*$. This means that A_{M^*} must happen (see Figure 7)

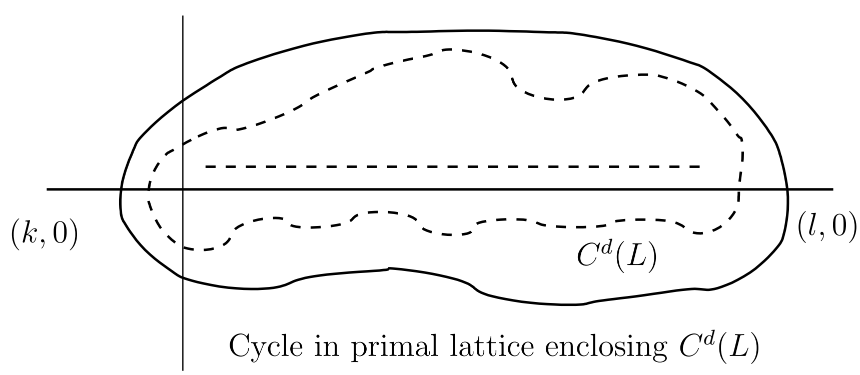

Figure 7: $C^d(L)$ enclosed by an open cycle

$$
\begin{aligned}
\mathrm{P}(|C^d(L)| < \infty) &\leq \mathrm{P}_p(A_{M^*}) < \tfrac{1}{2} \\
\Longrightarrow \mathrm{P}(|C^d(L)| = \infty) &> \tfrac{1}{2}
\end{aligned}
$$

Using pigeon hole principle we can argue that if the probability for M^* vertices is $> \frac{1}{2}$, then the probability of at least one vertex must be above the average, i.e.:

$$
\begin{aligned}
\Longrightarrow \mathrm{P}(\exists\, x \in L \text{ s.t. } |C^d(x)| = \infty) &> \frac{1}{2\, M^*} \\
\Longrightarrow \mathrm{P}(|C^d| = \infty) &> 0
\end{aligned}
$$

where C^d is the closed component in the dual lattice containing the origin. Because in an infinite lattice every vertex could be the origin, this proves Lemma 7.3.

Hence, we have shown that the critical probability p_c for bond percolation in 2 dimensions is $\frac{1}{2}$. ■

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