CSL857: Randomized algorithms	Instructor: Amitabha Bagchi
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Homework 1

Due: **24th January 2013, 11:59PM**

Q1. (Billingsley 1.3, page 15) Define a set A to be *trifling* if if for each $\epsilon > 0$ there exists a *finite* sequence of intervals I_k satisfying $A \subset \cup_k I_k$ and $\sum_k |I_k| < \epsilon$. This definition applies to all subsets of the real line, not just to all subsets of $(0, 1]$.

1. Show that a trifling set is negligible (as defined in Billingsley).
2. Show that the closure of a trifling set is trifling (where closure is defined as in analysis).
3. Find an example of a bounded negligible set that is not trifling.
4. Show that the closure of a negligible set may not be negligible.
5. Show that the finite unions of trifling sets are trifling but that this may not be true for countable unions.

Q2. (Billingsley 1.4, page 15) For $i = 0, 1, \ldots, r-1$, let $A_r(i)$ be the set of numbers in $(0, 1]$ whose nonterminating expansions in base r do *not* contain the digit i.

1. Show that $A_i(r)$ is trifling as defined in the previous question.
2. Find a trifling set A such that every point in $(0, 1]$ can be represented in the form $x + y$ where $x, y \in A$.

Q3. (Billingsley 2.3, page 33) Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be classes of sets in a common space Ω.

1. Suppose that $\mathcal{F}_1, \mathcal{F}_2, \ldots$ are fields satisfying $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Show that $\cup_{n=1}^{\infty} \mathcal{F}_n$ is an field.
2. Suppose that $\mathcal{F}_1, \mathcal{F}_2, \ldots$ are σ-fields satisfying $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Show by example that $\cup_{n=1}^{\infty} \mathcal{F}_n$ need not be a σ-field.

Q4. (Billingsley 4.5, page 65)

1. Show that $\lim_n \mathrm{P}(\liminf_k A_n \cap A_k^c) = 0$. Hint: Show that $\limsup_n \liminf_k A_n \cap A_k^c$ is empty.

2. Put $A^* = \limsup_n A_n$ and $A_* = \liminf_n A_n$. Show that $\mathrm{P}(A_n \setminus A^*) \to 0$ and $\mathrm{P}(A_* \setminus A_n) \to 0$.