CS105L: Discrete Structures
I semester, 2005-06

Solution to Tutorial Sheet 3, Problem 1

Instructor: Amitabha Bagchi

August 21, 2005

Important Clarification. In the lectures I said that the statement of Zorn's Lemma was: If every chain L of a partial order P has an upper bound *which lies in* L then P has a maximal element.

This is wrong, especially the highlighted part. Shubham brought this to my attention. The upper bound of a chain does not need to lie in the chain, it just has to lie in the partially ordered set. In other words the statement of Zorn's Lemma should be: If every chain of a partial order P has an upper bound that lies in P then P has a maximal element.

I apologise for this confusion and for the time and effort wasted as a result of my error.

The proof of the first tutorial problem is now relatively straightforward. I am putting it down here. The first problem in HW 3 can be solved using a similar construction.

Theorem 3.1 *Zorn's Lemma implies the Axiom of Choice.*

Proof. Given a family of sets $\mathcal{C}$ indexed by an index set I, i.e. the sets of $\mathcal{C}$ are called X_i where $i \in I$, we want to show that if Zorn's lemma holds, there is a choice function f such that $\forall i \in I : f(X_i) \in X_i$.

To do this we consider a partial order on the set P which is a set of partial functions from I to $\cup_{i \in I} X_i$ with the property that $e(i)$ is either not defined or $e(i) \in X_i$. Since a partial function from A to B can be thought of as a subset of $A \times B$, we consider the subset relation between the elements of P. In other words, for $g, h \in P$, $g \preceq h$ if $g(i) = h(i)$ whenever $g(i)$ is defined. This relation, $\preceq$, is a partial order on P.

To apply Zorn's lemma we have to satisfy ourselves that each chain in $(P, \preceq)$ has an upper bound in P. Note that this upper bound does not necessarily have to lie in the chain.

Consider a chain L in P. Now consider $h_L = \cup_{e_i \in L} e_i$. h_L is a partial function because if there is an i such that both (i, a) and (i, b) are in h_L where $a \neq b$, this means there are two elements in L which have differing images for i which would mean they are not comparable, hence contradicting the fact that L is a chain. Since h_L is a partial function it is definitely in P, and because it is the union of all the elements of L it is definitely an upper bound for L.

Since each chain has an upper bound, by Zorn's Lemma a maximal element exists in P. This maximal element, let us call it f, has to be a total function from I to $\cup_{i \in I} X_i$ because if it were not defined for any $i \in I$ it would be possible to find an element larger than it which was also defined for that i. By the property of all elements of P that they map $i \in I$ to an element of X_i, it is clear that F is a choice function for $\mathcal{C}$.

$\square$