CS105L: Discrete Structures
I semester, 2005-06

Homework # 6

Due before class on **Tuesday, October 4, 2005**

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1. A *vertex colouring* of a graph is an assignment of colours (or just natural numbers) to the vertices of a graph. A *colour class* in a coloured graph is a maximal set of vertices which have been assigned the same colour. A graph is said to be (k, d)-*colourable* if the vertices can be coloured with k colours in such a way that the vertices in each colour class induce a graph of maximum degree d. Prove the following theorem due to Lovász:

   Theorem 6.1 *(***Lovász, 1966***) For any k, any graph of maximum degree Δ with $|E|$ edges can be $(k, \lfloor \Delta/k \rfloor)$-coloured in time $O(\Delta|E|)$.*

2. Let us consider a multigraph G. To avoid confusion let us invent the term *edge-holder* which means a vertex pair and let us use the term *edge* to mean an actual edge between two vertices. In a multigraph the set of edge-holders is a set, and the set of edges is a multiset chosen from the set of edge-holders. In fact if there is no edge corresponding to a given edge-holder, then that edge-holder is of no interest to us. With this is mind let us try to prove a generalization of Menger's theorem (the edge version.)

   Given a pair of vertices (s, t) in a graph G, let us consider a maximal set of edge disjoint paths P between s and t (note, the paths of P are not edge-holder disjoint.) We call such a set of paths k-*balanced* if every edge holder occurs in less than $|P|/k$ paths of P. If there is an edge holder which occurs exactly $\lfloor |P|/k \rfloor$ times in P then it is known as a *critical edge*.

   (a) Prove, using Menger's theorem, that if a set of edge-disjoint paths P between s and t is k-balanced then it is possible to find a set of k edge-holder disjoint paths between s and t using the edges in P.

   (b) Prove that it is possible to find $\lfloor |P|/k \rfloor$ sets of k-edge holder disjoint paths between s and t using the edges in P. Note that each set of k paths has to be edge holder disjoint within itself but might share edge holders with other such sets.

   To prove the second part you will need the following lemma:

   Lemma 6.1 *Given a k-balanced path system with critical edges between two vertices s and t, there is a set of k-edge holder disjoint paths from s to t which pass through every critical edge.*