COL727: Rapid Mixing in Markov Chains
I semester, 2024-25
Minor III

Total Marks 100

Due: On Gradescope at 11:55PM, 21st October 2024

Instructions: 1. Please write legibly and make sure your scan is readable. 2. Remember to map problem numbers to pages of your pdf on Gradescope. 3. Be succinct in your answers but explain your calculations in words. 4. *In this exam* **page counts** *will be enforced. Use the number of pages allowed per question. Write/latex on standard A4 sheets in a reasonable sized font ($\geq$ 11pt for latex users). Only the given number of pages will be read in your scan*

Problem 1 ($10 + 30 = 40$ marks. Total page count: 3) *In this problem we will make a little excursion into random walks on infinite sets. Exercise 9.1 of LPW implies that the simple random walk on $\mathbb{Z}^d$ is* transient *whenever $d \geq 3$, i.e., the walk when started at $\mathbf{0} \in \mathbb{Z}^d$ returns to $\mathbf{0}$ a finite number of times with probability 1.*

Problem 1.1 (10 marks. Max page count: 1) *Let us fix $d = 3$. Show that for any finite $S \subset \mathbb{Z}^3$, the random walk begun at any $\boldsymbol{x} \in \mathbb{Z}^3$ visits S a finite number of times with probability 1. (Hint: The graph distance between any $\boldsymbol{x} \in \mathbb{Z}^3$ and any $S \subset \mathbb{Z}^3$ is finite.)*

Problem 1.2 (30 marks. Max page count: 2) *We define the Green's function for the walk on $\mathbb{Z}^3$ as*

$$G(\boldsymbol{x}, \boldsymbol{y}) = \mathbb{E}_{\boldsymbol{x}} \left\{ \sum_{t=0}^{\infty} \mathbf{1}_{\{X_t = \boldsymbol{y}\}} \right\}.$$

Also for any finite $S \subset \mathbb{Z}^3$ and $\boldsymbol{x} \in S$, we define the escape probability

$$\theta_S(\boldsymbol{x}) = \mathbb{P}_{\boldsymbol{x}} \left\{ \tau_S^+ = \infty \right\},$$

i.e., $\theta_S(\boldsymbol{x})$ is the probability that the random walk begun at $\boldsymbol{x} \in S$ never returns to S. Show that for any finite S and any $\boldsymbol{x} \notin S$

$$\mathbb{P}_{\boldsymbol{x}} \left\{ \tau_A < \infty \right\} = \sum_{y \in S} G(\boldsymbol{x}, \boldsymbol{y}) \theta_S(\boldsymbol{y}).$$

Problem 2 (20 marks. Max page count: 1) *Using the theory of Ch 9 of LPW prove that for any real number $w, x, y, z > 0$,*

$$\frac{(w + y)(x + z)}{w + x + y + z} \geq \frac{wx}{w + x} + \frac{yz}{y + z}.$$

Problem 3 (40 marks. Max page count: 2) *Consider the following graph that we can call the* barbell *on n vertices: It has two complete graphs with $n/3$ vertices and a path of length $n/3 + 1$ connecting one vertex in one of the complete graphs to one of the vertices in the other clique. Lower bound the mixing time of the lazy random walk on the barbell by by using the Normalized Laplacian method discussed in class. Also find an upper bound using any method.*