COL202: Discrete Mathematical Structures. I semester, 2022-23.
Amitabha Bagchi
Tutorial Sheet 7: Lattices.
13 October 2022

Important: The question marked with a ♠ is to be written on a sheet of paper and submitted to your TA within the first 10 minutes of the beginning of your tutorial session. Questions marked with a * are optional challenge problems and are not to be discussed in the tutorial.

Problem 1
Prove that any finite lattice is complete.

Problem 2
In [1] on page 122 it is stated that a finite poset is a lattice iff it has a greatest element and a least element. However this is not true. Prove that (i) a finite lattice has a greatest element and a least element and (ii) there is a finite poset with a greatest element and a least element which is not a lattice.

Problem 3
Prove that the meet and join of a complete lattice are commutative, associative and idempotent and have the absorption property, i.e., prove Proposition 4.2.2 of [1].

Problem 4
Given a set X, let $S \subseteq 2^X$ be a collection of subsets of X such that

1. $X \in S$, and
2. if $A_x \in S$ for all $x \in I$ where I is some index set, then $\cap_{x \in I} A_x$ is also in S.

Prove that $(S, \subseteq)$ is a complete lattice.

Problem 5
Write out the complete proof of step 2 of Tarski's Fixed Point Theorem (Theorem 4.2.6 of [1].)

Problem 6
Suppose that $(X, \preceq)$ is a lattice. For any $k > 1$, we define a relation $\preceq_k$ on X^k as follows: $(x_1, \ldots, x_k) \preceq_k (y_1, \ldots, y_k)$ if $\forall i \in [k] : x_i \preceq y_i$.

Problem 6.1 ♠
Prove that $(X^k, \preceq_k)$ is a lattice. Beging by showing that it is a poset.

Problem 6.2 *
If $(X, \preceq)$ is a complete lattice, is $(X^k, \preceq_k)$ also a complete lattice?

Problem 7
A lattice X is called *distributive* if for all $x, y, z \in X$,

- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, and
- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Give an example of a distributive lattice and a lattice which is not distributive.

Problem 8
Suppose that $(X, \leq)$ is a finite distributive lattice and there is an $a \in X$ such that a is minimal in $X \setminus \{\perp\}$. Let $S_1 = \{x \in X : a \not\leq x\}$ and $S_2 = \{x \in X : x = x' \vee a \text{ for some } x' \in S_1\}$. Show that S_1 and S_2 form distributive lattices.

Problem 9
Wisden decides to publish a book with exactly 100 pages containing biographies and photos of the top 100 cricketers of all time. The cricketers are ranked from 1 to 100. Although the text can be broken over multiple pages, each photo has to appear on a single page. Multiple photos can appear on the same page but the photos must appear in order of rank, i.e., the photo of cricketer ranked i must appear before the photo of cricketer ranked j whenever $i < j$.

Problem 9.1
Prove using Tarski's fixed point theorem that there is at least one cricketer whose photo appears on a page number equal to their rank (e.g. the photo of the 47th greatest cricketer appears on page 47).

Problem 9.2 *
Give another proof using induction.

References

[1] J. Gallier. Discrete Mathematics for Computer Science: Some Notes arXiv:0805.0585, 2008.