COL202: Discrete Mathematical Structures. I semester, 2022-23.
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Tutorial Sheet 2: Well-ordering and Inductions.
25 August 2022

Important: The question marked with a ♠ is to be written on a sheet of paper and submitted to your TA within the first 10 minutes of the beginning of your tutorial session.

Problem 1 ([1], Prob 1, pp 126)

Use the Well Ordering Principle to prove that

$$\frac{2}{3} + \frac{2}{9} + \cdots + \frac{2}{3^n} = 1 - \left(\frac{1}{3}\right)^n,$$

for all $n \geq 1$.

Problem 2 ([2], Problem 2.5)

Use the Well Ordering Principle to prove that there is no positive integer solution for

$$4a^3 + 2b^3 = c^3.$$

Problem 3

Let us assume that the postal department has only two denominations of stamps: Rs 3 and Rs 5. We call a number n *postal* if we can create postage worth Rs n using the two denominations that are available. Prove using the Well Ordering Principle that every $n \geq 8$ is postal.

Problem 4 ([1], Prob 13, pp 127)

Use mathematical induction to prove that the number of subsets of a set of size n is 2^n.

Problem 5

We are given an array $A[n]$ containing 0s and 1s only. We want to sort it so that all the 0s appear before all the 1s, e.g. if $A = [1, 0, 0, 1, 0, 0]$ our output should be $A = [0, 0, 0, 0, 1, 1]$. Prove by induction that the procedure "Sort 0-1" correctly achieves this.

Set $p_0 \leftarrow 0$, $p_1 \leftarrow n - 1$.
while $p_0 < p_1$ **do**
    while $p_0 < n$ and $A[p_0] = 0$ **do**
        $p_0 \leftarrow p_0 + 1$
    end while
    while $p_1 > -1$ and $A[p_1] = 1$ **do**
        $p_1 \leftarrow p_1 - 1$
    end while
    if $p_0 < p_1$ **then**
        Swap $A[p_0], A[p_1]$.
    end if
end while

Problem 6

Suppose we are given a linked list with integers stored in each node and suppose that the linked list is maintained in sorted order. Write an algorithm for inserting a new element ℓ in the linked list. Prove the correctness of your algorithm using mathematical induction on the number of elements in the linked list.

Problem 7 ♠

Integer trees are a recursively defined data type. Every tree is either an empty tree, we denote it EmptyTree or a tuple of the form (ℓ, T_1, T_2) where ℓ is an integer and T_1 and T_2 are trees which we call the left and right subtrees respectively. Write an algorithm for finding the minimum integer in the tree. Assume for simplicity that all integers stored are non-negative. Prove the correctness of your algorithm using mathematical induction.

Problem 8 ([1], Prob 14, pp 127)
Prove that the strong principle of mathematical induction implies the weak principle of mathematical induction, i.e., if we accept that the deduction rule of the strong principle is sound then the deduction rule of the weak principle is also sound.

References

[1] K. Bogart, S. Drysdale, C. Stein Discrete Math for Computer Science Students. 2005.

[2] E. Lehman, F. T. Leighton, A. R. Meyer Mathematics for Computer Science, 2018 revision 2018.