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This is an elaboration of a proof from Graph Theory by Reinhard Diestel.

Diestel Lemma 1.5.5 (ii): Let T be a normal tree in G. If $S \subseteq V(T) = V(G)$ and S is down-closed, then the components of $G - S$ are spanned by the sets $\lfloor x \rfloor$ with x minimal in $T - S$.

Proof: We begin by taking any component C of $G - S$, and a vertex x that is a minimal element of $V(C)$. We first show that x is the only minimal element of $V(C)$. Suppose, there is another minimal element x' of $V(C)$, then any x–x' path in C would contain a vertex below both (from Lemma 1.5.5(i)), contradicting the assumption of their minimality.

Now, we shall show that the up closure of x spans the component C. Diestel says "As every vertex of C lies above some minimal element of $V(C)$, it lies above x." We shall also show its converse by stating that every vertex $y \in \lfloor x \rfloor$ lies in the component C, because if $y \in S$, then since S is down closed, $x \in S$ will also hold, which is a contradiction.

From the above arguments, we have that $V(C) = \lfloor x \rfloor$.

Now, we have established that the components of $G - S$ are spanned by the upclosures of their minimal elements. To prove the lemma, we must show that the set of minimal elements of components of $G - S$ is the same as the set of minimal elements of $T - S$.

We begin by showing that x is a minimal element of $T-S$. The set of all vertices lying below x in T form a chain which can be represented as $\lceil t \rceil$, for a vertex t which is a neighbour of x. Since x is minimal in $V(C)$ and t is a neighbour of x in T, t must not belong to $V(C)$. Hence $t \in S$, and consequently, all vertices in $\lceil t \rceil$ belong to S, since S is down closed.

The converse is straightforward, as if x is a minimal element of $T-S$, it is also a minimal element of every subset of $T-S$ that it belongs to, and hence is a minimal element of the component of $G-S$ that it belongs to.

From the above two arguments, we shown that the set of minimal vertices of **$T-S$** is the same as the set of minimal vertices of components of $G-S$.

Combining this with the result that the components of $G-S$ are spanned by $\lfloor x \rfloor$, where x is the minimal element of the component, we have shown that "the components of $G-S$ are spanned by the sets $\lfloor x \rfloor$ with x minimal in $T-S$", since the set of minimal elements of the components of $G-S$ is the same as the set of minimal elements of $T-S$.