COL202: Discrete Mathematical Structures. I semester, 2020-21.
Amitabha Bagchi
Tutorial Sheet 8: Basic counting.
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Important: The question marked with a ♠ is to be submitted via gradescope by 11:59PM on the day that you have your tutorial.

Problem 1
Suppose you have n men and n women and you have to seat them around a circular table so that men and women sit alternately. How many ways can you seat them?

Problem 2
Suppose there are n tennis players. How many ways can we pair them up so that everyone has a partner to play a singles match?

Problem 3
Given a set A of size m and set B of size n count the number of

1. Relations from A to B.
2. Total functions from A to B.
3. Partial functions from A to B.
4. Surjections from A to B (assume $m \geq n$). These could be partial or total functions.
5. Injections from A to B. These could be partial or total. State whatever you assume about m and n.

Problem 4
Given a plane with integer points of the type (x, y) where both x and y are integers, we define a *lattice path* from (x_1, y_1) to (x_2, y_2) to be a set of line segments that go from a point (i, j) to $(i+1, j)$ or $(i, j+1)$, i.e., all steps in the path either move right or up. Count the number of lattice paths between $(0, 0)$ and (m, n)?

Problem 5
A lattice path from $(0, 0)$ to (n, n) is called a *Catalan path* if it only visits points (x, y) such that $y \leq x$. Count the number of Catalan paths between $(0, 0)$ and (n, n). (**Hint.** Argue that every lattice path that is *not* a Catalan path must touch or cross the line $y = x + 1$. Find a bijection between the set of lattice paths that touch or cross the line $y = x + 1$ and the set of lattice paths between (-1,1) and (n, n).)

Problem 6 ♠
We say that a function π is a *derangement of size* n if it is a bijection from $\{1, \ldots, n\}$ to itself (i.e., it is a permutation) and it has no fixed points, i.e., $\forall i : \pi(i) \neq i$. Count the number of derangements of size n. The solution you submit must use Inclusion-Exclusion. Separately also try to see if you can solve the problem without the use of Inclusion-Exclusion, but you don't have to submit the other solution(s).

Problem 7
Prove the following identities regarding binomial coefficients by making counting arguments. Give as many different arguments as possible

Problem 7.1

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}.$$

Problem 7.2

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{j}\binom{n-j}{k}.$$

Problem 7.3

$$\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}.$$

Problem 8
Show that any finite connected graph on n vertices must have two vertices with the same degree.

Problem 9
Let A be any set of 20 numbers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A which sum to 104.

Problem 10
Consider any five points in the interior of a square of side 1. Show that there are two points which are at most $1/\sqrt{2}$ units apart. Is this this the best possible bound i.e. is there a placement of five points such that the maximum interpoint distance is exactly $1/\sqrt{2}$.