COL202: Discrete Mathematical Structures. I semester, 2020-21.
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Tutorial Sheet 6: Relations and Partial Orders.
12 November 2020

Important: The question marked with a ♠ is to be submitted via gradescope by 11:59PM on the day that you have your tutorial.

Problem 1 [1]
Prove that for binary relations $\mathcal{R}, \mathcal{R}'$ from A to B and $\mathcal{S}, \mathcal{S}'$ from B to C, if $\mathcal{R} \subseteq \mathcal{R}'$ and $\mathcal{S} \subseteq \mathcal{S}'$ then $\mathcal{R} \circ \mathcal{S} \subseteq \mathcal{R}' \circ \mathcal{S}'$.

Problem 2 [1]
Given $\mathcal{R} \subseteq A \times B$ and $\mathcal{S}, \mathcal{T} \subseteq B \times C$, prove or find an example that disproves

1. $\mathcal{R} \circ (\mathcal{S} \cup \mathcal{T}) = (\mathcal{R} \circ \mathcal{S}) \cup (\mathcal{R} \circ \mathcal{T})$
2. $\mathcal{R} \circ (\mathcal{S} \cap \mathcal{T}) = (\mathcal{R} \circ \mathcal{S}) \cap (\mathcal{R} \circ \mathcal{T})$
3. $\mathcal{R} \circ (\mathcal{S} \setminus \mathcal{T}) = (\mathcal{R} \circ \mathcal{S}) \setminus (\mathcal{R} \circ \mathcal{T})$

Problem 3 [1]
Show that a relation $\mathcal{R}$ on a set A is

1. antisymmetric if and only if $\mathcal{R} \cap \mathcal{R}^{-1} \subseteq \mathcal{I}_A$.
2. transitive if and only $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$.
3. connected if and only if $(A \times A) \setminus \mathcal{I}_A \subseteq \mathcal{R} \cup \mathcal{R}^{-1}$.

Problem 4 [1]
Given a relation $\mathcal{R}$ on a set A, we define a sequence of relations as follows: we say $\mathcal{R}_0 = \mathcal{I}_A$ and $\mathcal{R}_{i+1} = \mathcal{R}_i \circ \mathcal{R}$. Based on this we define the *reflexive transitive closure* of $\mathcal{R}$ as

$$\mathcal{R}^* = \bigcup_{i \geq 0} \mathcal{R}_i.$$

1. Prove or disprove that $\mathcal{S} = \mathcal{R}^* \cup (\mathcal{R}^*)^{-1}$ and $\mathcal{T} = (\mathcal{R} \cup \mathcal{R}^{-1})^*$ are both equivalence relations.
2. Prove or disprove $\mathcal{S} = \mathcal{T}$.

Problem 5 [1]
Consider any preorder $\mathcal{R}$ on A. For each $a \in A$ let $[a]_\mathcal{R} = \{b \in A : a\mathcal{R}b \wedge b\mathcal{R}a\}$. Now let $B = \{[a]_\mathcal{R} : a \in A\}$. Define a relation $\mathcal{S} \subseteq B \times B$ as follows: $[a]_\mathcal{R}\mathcal{S}[b]_\mathcal{R}$ whenever $a\mathcal{R}b$. Show that $\mathcal{S}$ is a partial order.

Problem 6
Suppose we have a set S and a partially ordered set $(T, \preceq_T)$, let $\mathcal{F}$ be the set of functions $f : S \to T$, i.e., all the functions from S to T. We define a relation, $\preceq$, on $\mathcal{F}$ as follows: $f \preceq g$ if $f(x) \preceq_T g(x)$ for all $x \in S$. Show that $\preceq$ is a partial order on $\mathcal{F}$.

Problem 7 ♠
Let $(S, \preceq_S)$ and $(T, \preceq_T)$ be two posets defined on disjoint sets S, T. The *linear sum* $S \oplus T$ of the two posets is $(S \cup T, \preceq)$ where for $x, y \in S \cup T$ we say $x \preceq y$ if either $x \preceq_S y$ or $x \preceq_T y$ or if $x \in S$ and $y \in T$. Show that $\preceq$ is a partial order on $S \cup T$. Draw an example of a graph for which a normal spanning tree's tree order can be represented as the linear sum of two tree orders.

Problem 8
Two partially ordered sets $(S, \preceq_S)$ and $(T, \preceq_T)$ are said to be *isomorphic* if there exists a bijection $f : S \to T$ such that $x \preceq_S y$ if and only if $f(x) \preceq_T f(y)$ for all $x, y \in S$. The function f is called an *isomorphism.* Also a function $f : S \to T$ is said to be *increasing* if $x \preceq_S y$ implies $f(x) \preceq_T f(y)$ for all $x, y \in S$. A function $f : S \to T$ is said to be *strictly* increasing iff for $x \neq y$, $x \preceq_S y$ implies $f(x) \preceq_T f(y)$ and $f(x) \neq f(y)$ (this could also be denoted $f(x) \prec_T f(y)$).

Show by example that an increasing function need not be an isomorphism.

Problem 9
Suppose $(S, \preceq_S)$ and $(T, \preceq_T)$ are *isomorphic* and $f : S \to T$ is an isomorphism between them. Show that f and f^{-1} are both strictly increasing functions.

References

[1] S. Arun-Kumar, Lecture notes for *Introduction to Logic for Computer Science.*, IIT Delhi, 2002. http://www.cse.iitd.ernet.in/∼sak/courses/ilcs/logic.pdf