Recursion

Recursion

- A process by which a function calls itself repeatedly
  - Either directly.
    - X calls X
  - Or cyclically in a chain.
    - X calls Y, and Y calls X

- Used for repetitive computations in which each action is stated in terms of a previous result

  fact(n) = n * fact (n-1)

Contd.

- For a problem to be written in recursive form, two conditions are to be satisfied:
  - It should be possible to express the problem in recursive form
    - Solution of the problem in terms of solution of the same problem on smaller sized data
  - The problem statement must include a stopping condition

Stopping condition

$$\begin{aligned} fact(n) &= 1, && \text{if } n = 0 \\ &= n * fact(n-1), && \text{if } n > 0 \end{aligned}$$

Recursive definition

Examples:

- Factorial:

  fact(0) = 1

  fact(n) = n * fact(n-1), if n > 0

- GCD:

  gcd (m, m) = m

  gcd (m, n) = gcd (m%n, n), if m > n

  gcd (m, n) = gcd (n, n%m), if m < n

- Fibonacci series (1,1,2,3,5,8,13,21,….)

  fib (0) = 1

  fib (1) = 1

  fib (n) = fib (n-1) + fib (n-2), if n > 1

Factorial

```
long  int  fact (int n)
{
    if   (n == 1)
        return (1);
    else
        return  (n * fact(n-1));
}
```

Factorial Execution

```
long  int  fact (int n)
{
   if   (n = = 1) return (1);
   else return  (n * fact(n-1));
}
```

Factorial Execution

fact(4)

```
long  int  fact (int n)
{
    if   (n = = 1) return (1);
    else return  (n * fact(n-1));
}
```

Factorial Execution

fact(4)

```
if   (4 = = 1) return (1);
else return  (4 * fact(3));
```

```
long  int  fact (int n)
{
   if   (n = = 1) return (1);
   else return  (n * fact(n-1));
}
```

Factorial Execution

fact(4)

```
if   (4 = = 1) return (1);
else return  (4 * fact(3));
```

```
if   (3 = = 1) return (1);
else return  (3 * fact(2));
```

```
long  int  fact (int n)
{
   if   (n = = 1) return (1);
   else return  (n * fact(n-1));
}
```

Factorial Execution

fact(4)

```
if   (4 = = 1) return (1);
else return  (4 * fact(3));
```

```
if   (3 = = 1) return (1);
else return  (3 * fact(2));
```

```
if   (2 = = 1) return (1);
else return  (2 * fact(1));
```

```
long  int  fact (int n)
{
   if   (n = = 1) return (1);
   else return  (n * fact(n-1));
}
```

Factorial Execution

```
fact(4)
↓
if   (4 = = 1) return (1);
else return  (4 * fact(3));
                  ↓
                  if   (3 = = 1) return (1);
                  else return  (3 * fact(2));
                                    ↓
                                    if   (2 = = 1) return (1);
                                    else return  (2 * fact(1));
                                                      ↓
                                                      if   (1 = = 1) return (1);
```

```
long  int  fact (int n)
{
   if   (n = = 1) return (1);
   else return  (n * fact(n-1));
}
```

Factorial Execution

fact(4)

```
if   (4 = = 1) return (1);
else return  (4 * fact(3));

    if   (3 = = 1) return (1);
    else return  (3 * fact(2));

        if   (2 = = 1) return (1);
        else return  (2 * fact(1));      1

            if   (1 = = 1) return (1);
```

```
long  int  fact (int n)
{
   if   (n = = 1) return (1);
   else return  (n * fact(n-1));
}
```

Factorial Execution

fact(4)

```
if   (4 = = 1) return (1);
else return  (4 * fact(3));
```

```
if   (3 = = 1) return (1);
else return  (3 * fact(2));
```

2

```
if   (2 = = 1) return (1);
else return  (2 * fact(1));
```

1

```
if   (1 = = 1) return (1);
```

```
long  int  fact (int n)
{
   if   (n = = 1) return (1);
   else return  (n * fact(n-1));
}
```

Factorial Execution

fact(4)

```
if   (4 = = 1) return (1);
else return  (4 * fact(3));
```

```
if   (3 = = 1) return (1);
else return  (3 * fact(2));
```

2

```
if   (2 = = 1) return (1);
else return  (2 * fact(1));
```

1

```
if   (1 = = 1) return (1);
```

```
long  int  fact (int n)
{
   if   (n = = 1) return (1);
   else return  (n * fact(n-1));
}
```

Factorial Execution

fact(4)

```
if   (4 = = 1) return (1);
else return  (4 * fact(3));        6

    if   (3 = = 1) return (1);
    else return  (3 * fact(2));    2

        if   (2 = = 1) return (1);
        else return  (2 * fact(1));    1

            if   (1 = = 1) return (1);
```

```
long  int  fact (int n)
{
   if   (n = = 1) return (1);
   else return  (n * fact(n-1));
}
```

Factorial Execution

```
fact(4)                                              24

  if   (4 = = 1) return (1);
  else return  (4 * fact(3));                                    6

            if   (3 = = 1) return (1);
            else return  (3 * fact(2));                                   2

                      if   (2 = = 1) return (1);
                      else return  (2 * fact(1));                  1

                              if   (1 = = 1) return (1);
```

```
long  int  fact (int n)
{
   if   (n = = 1) return (1);
   else return  (n * fact(n-1));
}
```

Look at the variable addresses (a slightly different program) !

```
void main()
{
    int  x,y;
    scanf("%d",&x);
    y = fact(x);
    printf  ("M: x= %d, y = %d\n", x,y);
}

int fact(int data)
{ int val = 1;
  printf("F: data = %d, &data = %u \n
    &val = %u\n", data, &data, &val);
  if (data>1) val = data*fact(data-1);
    return val;
}
```

Output

```
4
F: data = 4, &data = 3221224528
 &val = 3221224516
F: data = 3, &data = 3221224480
 &val = 3221224468
F: data = 2, &data = 3221224432
 &val = 3221224420
F: data = 1, &data = 3221224384
 &val = 3221224372
M: x= 4, y = 24
```

Fibonacci Numbers

Fibonacci recurrence:

fib(n) = 1 if n = 0 or 1;

= fib(n – 2) + fib(n – 1)

otherwise;

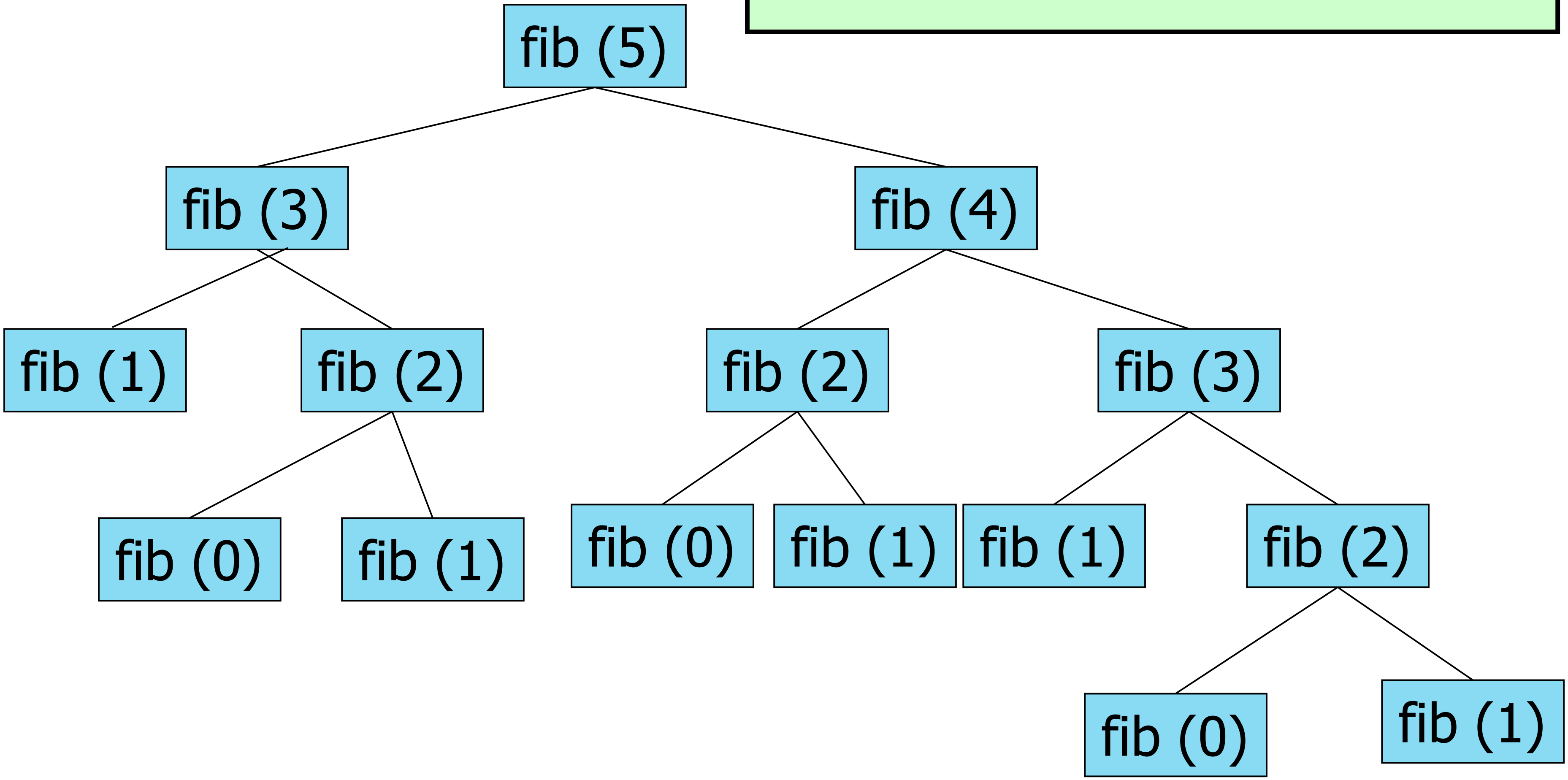
```
int fib (int n){
    if (n == 0 or n == 1)
        return 1;     [BASE]
    return fib(n-2) + fib(n-1) ;
                          [Recursive]
 }
```

```
int fib (int n)    {
   if (n == 0 || n == 1)
          return 1;
   return fib(n-2) + fib(n-1) ;
}
```

Fibonacci recurrence:

fib(n) = 1 if n = 0 or 1;

= fib(n – 2) + fib(n – 1) otherwise;

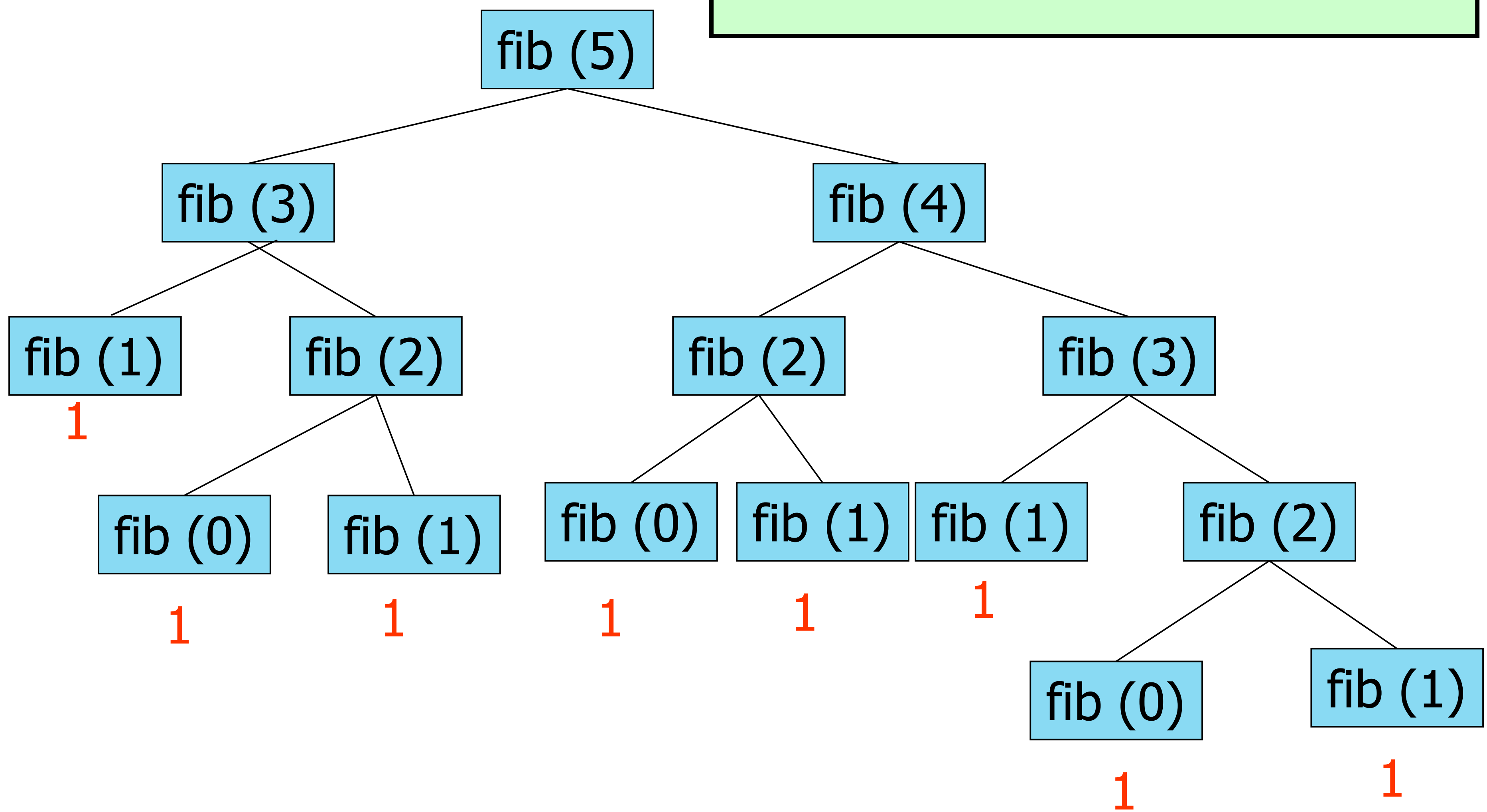


```
int fib (int n)    {
   if (n == 0 || n == 1)
          return 1;
   return fib(n-2) + fib(n-1) ;
}
```

Fibonacci recurrence:

fib(n) = 1 if n = 0 or 1;

= fib(n – 2) + fib(n – 1) otherwise;

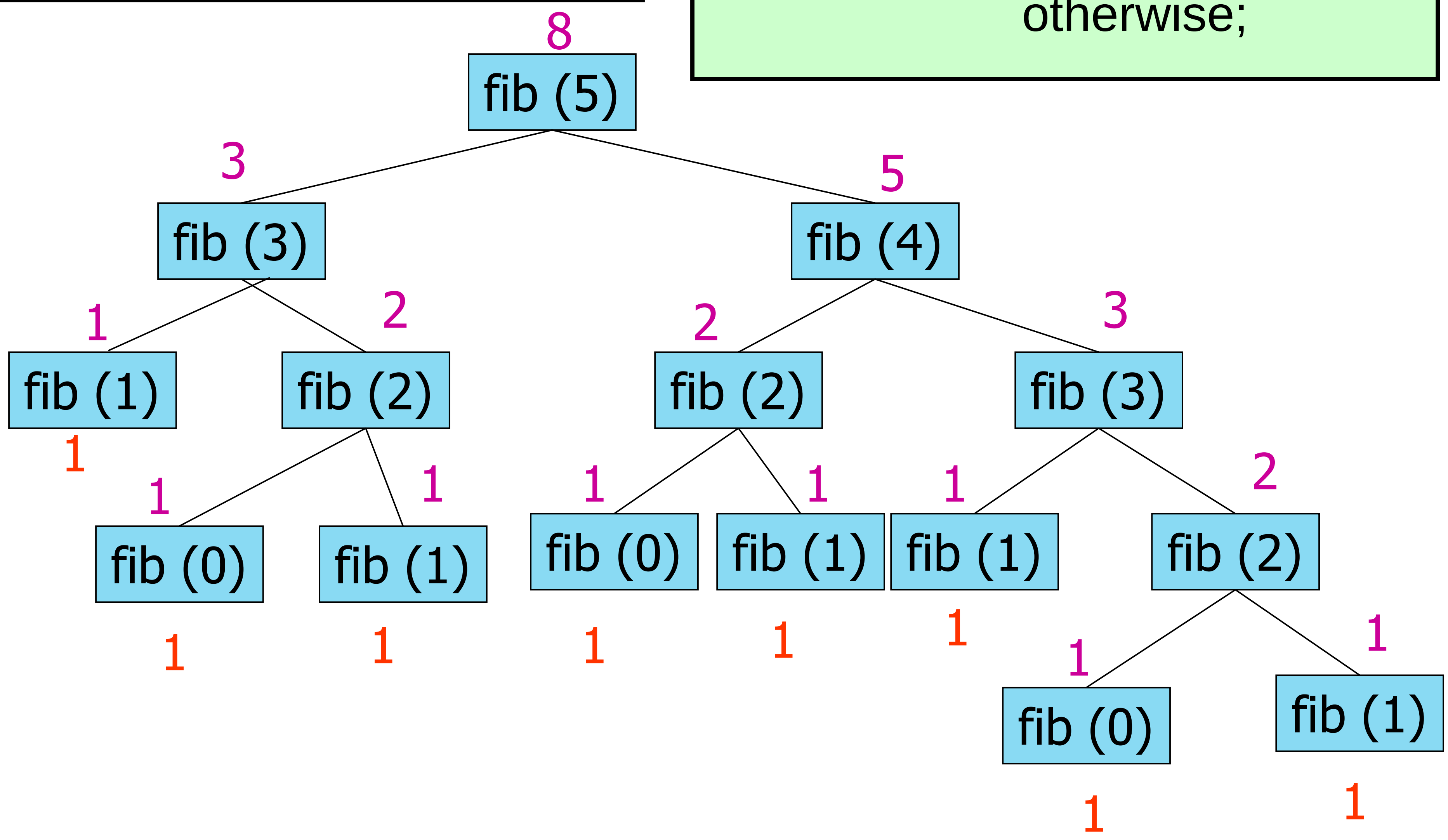


```
int fib (int n)    {
   if (n==0 || n==1)
         return 1;
   return fib(n-2) + fib(n-1) ;
}
```

Fibonacci recurrence:

fib(n) = 1 if n = 0 or 1;

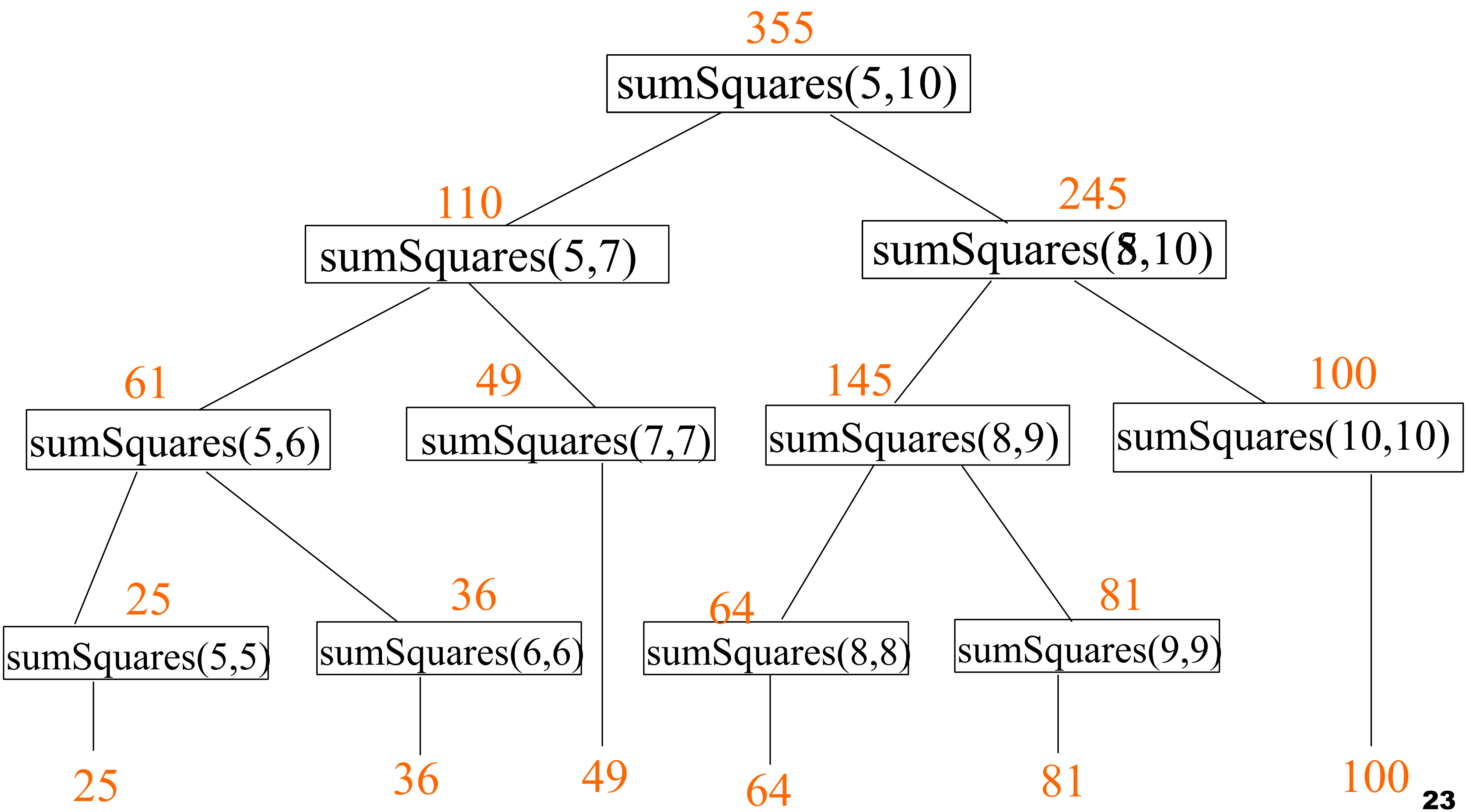
= fib(n – 2) + fib(n – 1) otherwise;



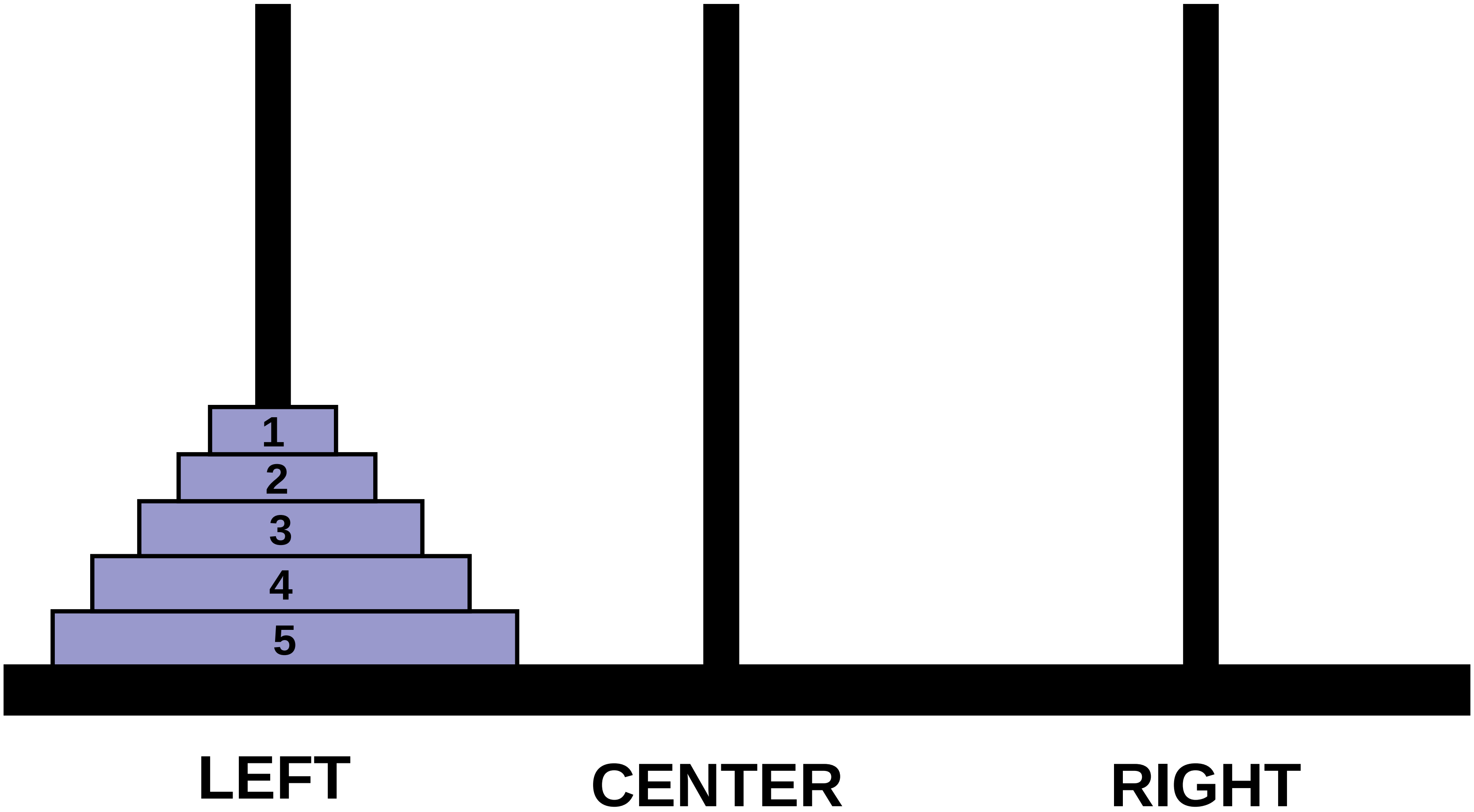
Sum of Squares

```
int sumSquares (int m, int n)
{
    int middle ;
    if (m == n) return m*m;
    else
    {
        middle = (m+n)/2;
        return sumSquares(m,middle)
                    + sumSquares(middle+1,n);
     }
}
```

Annotated Call Tree



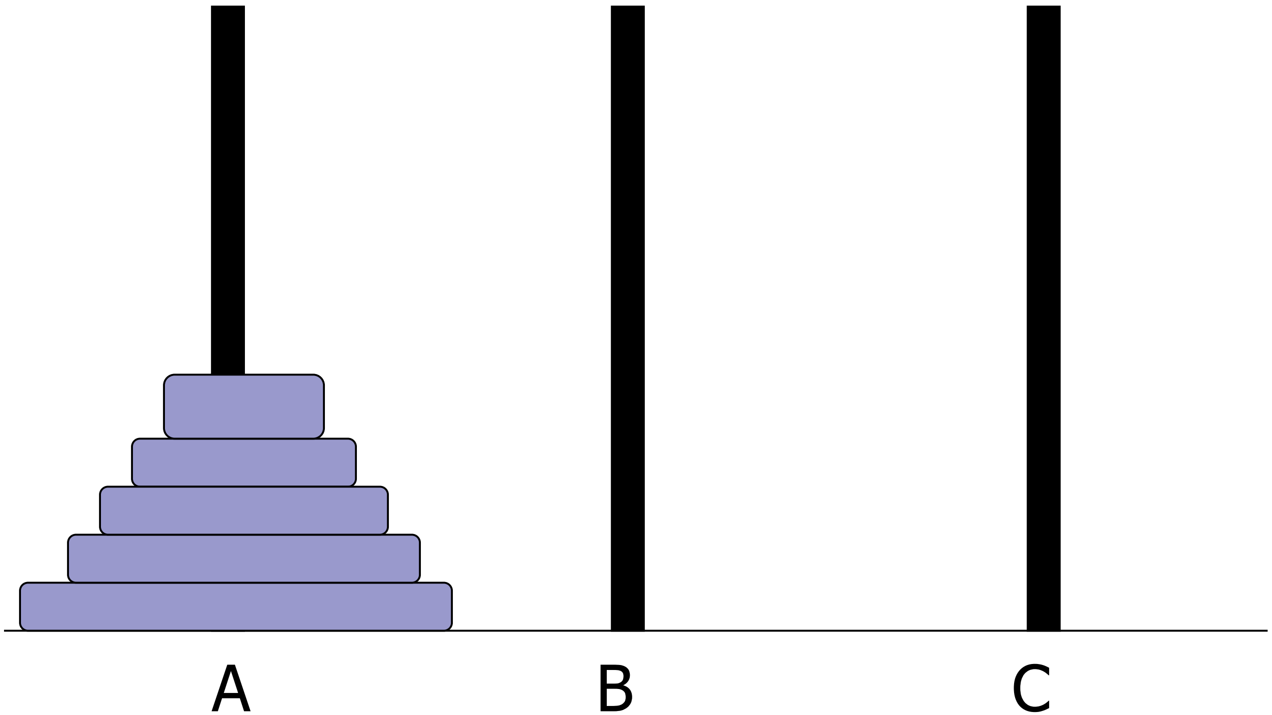
Towers of Hanoi Problem



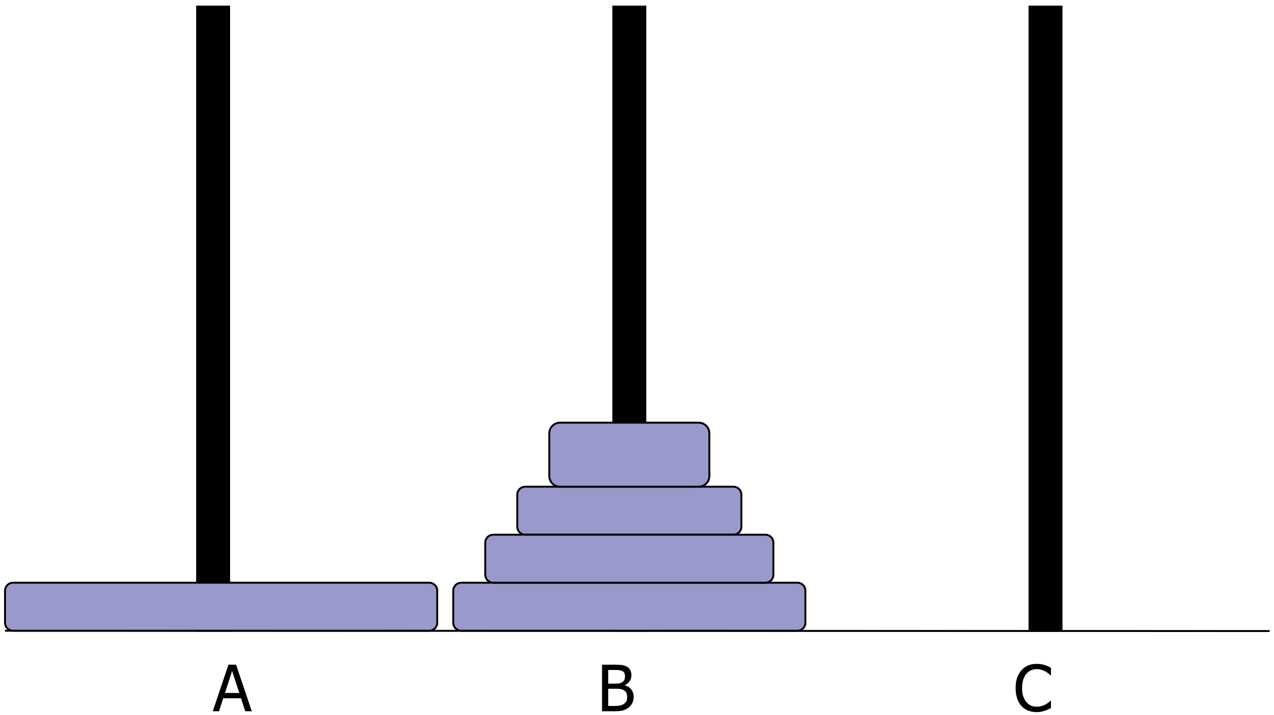
- Initially all the disks are stacked on the LEFT pole
- Required to transfer all the disks to the RIGHT pole
  - Only one disk can be moved at a time.
  - A larger disk cannot be placed on a smaller disk
- CENTER pole is used for temporary storage of disks

- Recursive statement of the general problem of n disks
  - Step 1:
    - Move the top (n-1) disks from LEFT to CENTER
  - Step 2:
    - Move the largest disk from LEFT to RIGHT
  - Step 3:
    - Move the (n-1) disks from CENTER to RIGHT

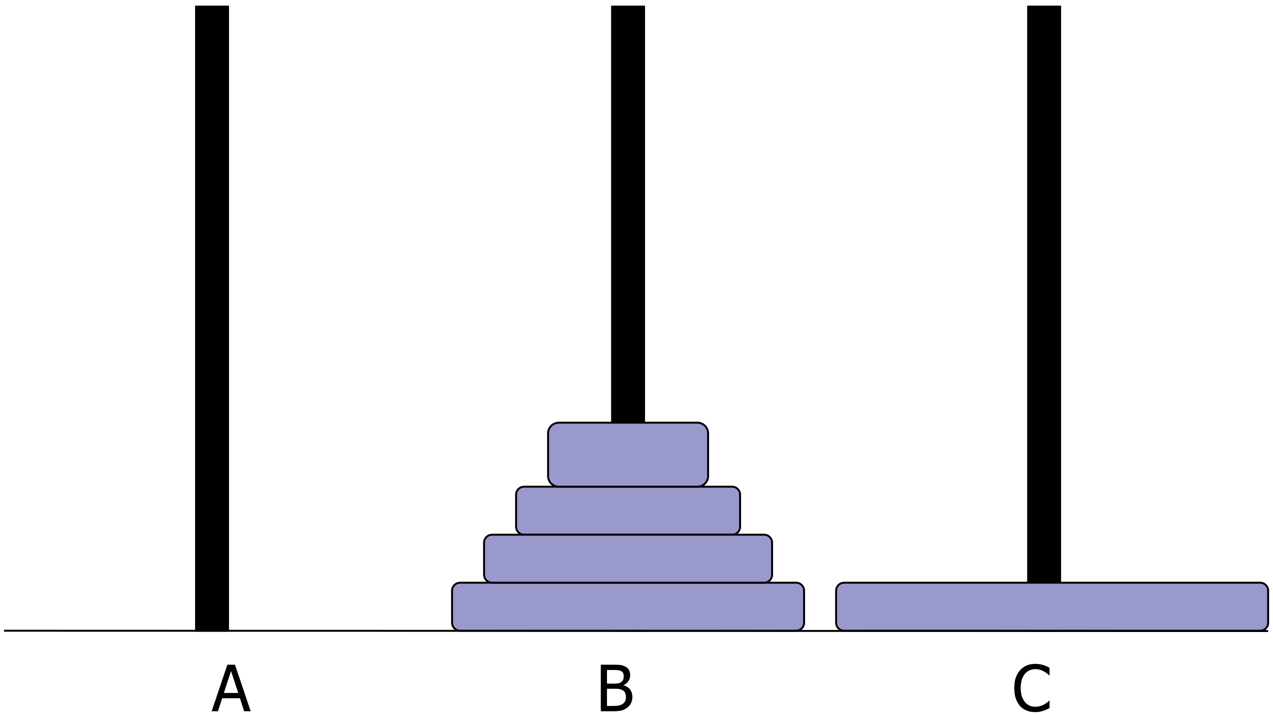
Tower of Hanoi



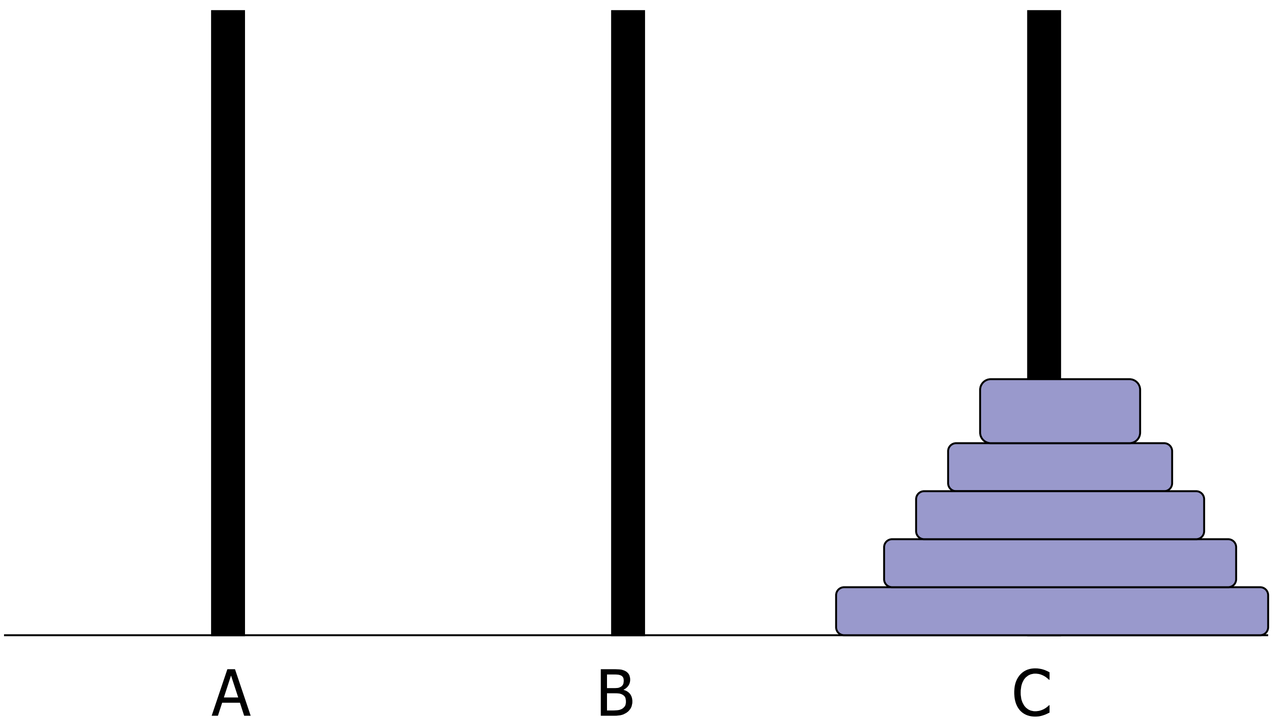
Tower of Hanoi



Tower of Hanoi



Tower of Hanoi



Towers of Hanoi function

```
void towers (int n, char from, char to, char aux)
{
 /* Base Condition */
 if (n==1)   {
          printf ("Disk 1 : %c → &c \n", from, to) ;
          return ;
     }
    /* Recursive Condition */
        towers (n-1, from, aux, to) ;
        ……………………….
        ……………………….
}
```

Towers of Hanoi function

```
void towers (int n, char from, char to, char aux)
{
 /* Base Condition */
 if (n==1)   {
          printf ("Disk 1 : %c →  &c \n", from, to) ;
          return ;
     }
    /* Recursive Condition */
        towers (n-1, from, aux, to) ;
        printf ("Disk %d : %c → %c\n", n, from, to) ;
        ……………………….
}
```

Towers of Hanoi function

```
void towers (int n, char from, char to, char aux)
{
 /* Base Condition */
 if (n==1)   {
           printf ("Disk 1 : %c → %c \n", from, to) ;
           return ;
      }
     /* Recursive Condition */
         towers (n-1, from, aux, to) ;
         printf ("Disk %d : %c → %c\n", n, from, to) ;
         towers (n-1, aux, to, from) ;
}
```

TOH runs

```
void towers(int n, char from, char to, char aux)
{ if (n==1)
 {  printf ("Disk 1 : %c -> %c \n", from, to) ;
    return ;
 }
   towers (n-1, from, aux, to) ;
   printf ("Disk %d : %c -> %c\n", n, from, to) ;
   towers (n-1, aux, to, from) ;
}
void main()
{ int n;
  scanf("%d", &n);
  towers(n,'A',‘C',‘B');
}
```

Output

```
3
Disk 1 : A -> C
Disk 2 : A -> B
Disk 1 : C -> B
Disk 3 : A -> C
Disk 1 : B -> A
Disk 2 : B -> C
Disk 1 : A -> C
```

More TOH runs

```
void towers(int n, char from, char to, char aux)
{ if (n==1)
 {  printf ("Disk 1 : %c -> %c \n", from, to) ;
    return ;
 }
   towers (n-1, from, aux, to) ;
   printf ("Disk %d : %c -> %c\n", n, from, to) ;
   towers (n-1, aux, to, from) ;
}
void main()
{ int n;
  scanf("%d", &n);
  towers(n,'A',‘C',‘B');
}
```

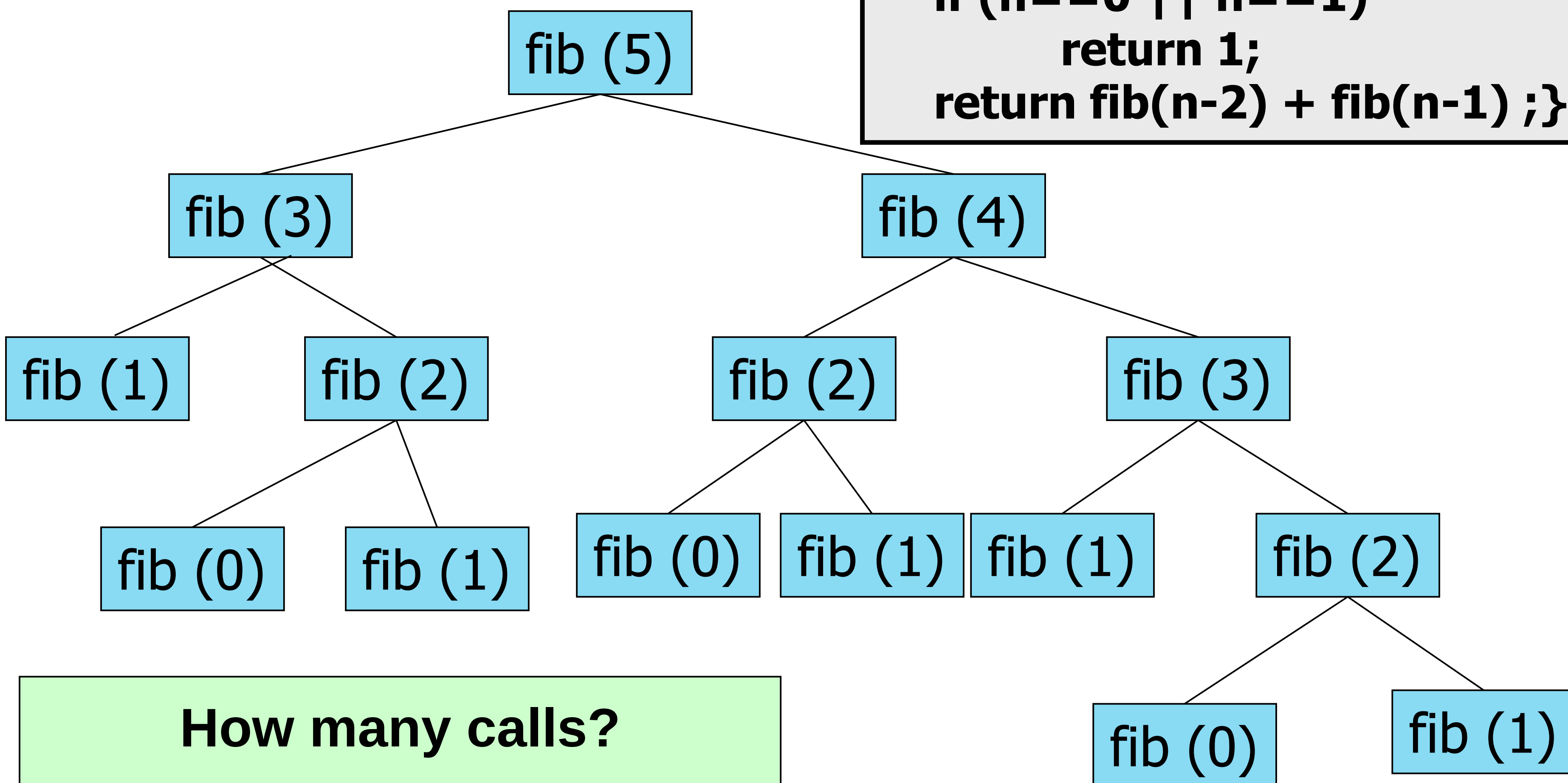
Output

```
4
Disk 1 : A -> B
Disk 2 : A -> C
Disk 1 : B -> C
Disk 3 : A -> B
Disk 1 : C -> A
Disk 2 : C -> B
Disk 1 : A -> B
Disk 4 : A -> C
Disk 1 : B -> C
Disk 2 : B -> A
Disk 1 : C -> A
Disk 3 : B -> C
Disk 1 : A -> B
Disk 2 : A -> C
Disk 1 : B -> C
```

Relook at recursive Fibonacci:

Not efficient !! Same sub-problem solved many times.

```
int fib (int n)    {
   if (n==0 || n==1)
          return 1;
   return fib(n-2) + fib(n-1) ;}
```



How many calls?

How many additions?

Iterative Fib

```
int fib( int n)
{ int i=2, res=1, m1=1, m2=1;
  if (n ==0 || n ==1) return res;
  for (  ; i<=n; i++)
  { res = m1 + m2;
    m2 = m1;
    m1 = res;
  }
  return res;
}
void main()
{ int n;
  scanf("%d", &n);
  printf(" Fib(%d) = %d \n", n, fib(n));
}
```

Much Less Computation here!
(How many additions?)

An efficient recursive Fib

```
int Fib(int m1, int m2, int n, int i)
{
  int res;
  if (n == i)
     res = m1+ m2;
  else
     res = Fib(m1+m2, m1, n, i+1);
  return res;
}
```

```
int Fib ( int, int, int, int);

void main()
{
  int n;
  scanf("%d", &n);
  if (n == 0 || n ==1)
     printf("F(%d) = %d \n", n, 1);
  else
     printf("F(%d) = %d \n", n, Fib(1,1,n,2));
}
```

Much Less Computation here!
(How many calls/additions?)

Run

```
int Fib ( int, int, int, int);
void main()
{ int n;
  scanf("%d", &n);
  if (n == 0 || n ==1)  printf("F(%d) = %d \n", n, 1);
  else  printf("F(%d) = %d \n", n, Fib(1,1,n,2));
}
int Fib(int m1, int m2, int n, int i)
{ int res;
  printf("F: m1=%d, m2=%d, n=%d, i=%d\n",
                                        m1,m2,n,i);
 if (n == i)
    res = m1+ m2;
  else
    res = Fib(m1+m2, m1, n, i+1);
 return res;
}
```

Output

```
$ ./a.out
3
F: m1=1, m2=1, n=3, i=2
F: m1=2, m2=1, n=3, i=3
F(3) = 3


$ ./a.out
5
F: m1=1, m2=1, n=5, i=2
F: m1=2, m2=1, n=5, i=3
F: m1=3, m2=2, n=5, i=4
F: m1=5, m2=3, n=5, i=5
F(5) = 8
```

Static Variables

```
int Fib (int, int);

void main()
{
    int n;
    scanf("%d", &n);
    if (n == 0 || n ==1)
       printf("F(%d) = %d \n", n, 1);
    else
       printf("F(%d) = %d \n", n,
   Fib(n,2));
}
```

```
int Fib(int n, int i)
{
    static int m1, m2;
    int res, temp;
    if (i==2) {m1 =1; m2=1;}
    if (n == i)  res = m1+ m2;
    else
     {   temp = m1;
         m1 = m1+m2;
         m2 = temp;
        res = Fib(n, i+1);
     }
    return res;
}
```

Static variables remain in existence rather than coming and going each time a function is activated

Static Variables: See the addresses!

```
int Fib(int n, int i)
{
  static int m1, m2;
  int res, temp;
  if (i==2) {m1 =1; m2=1;}
  printf("F: m1=%d, m2=%d, n=%d,
                    i=%d\n", m1,m2,n,i);
  printf("F: &m1=%u, &m2=%u\n",
                                &m1,&m2);
  printf("F: &res=%u, &temp=%u\n",
                       &res,&temp);
  if (n == i)  res = m1+ m2;
  else {  temp = m1;  m1 = m1+m2;
        m2 = temp;
        res = Fib(n, i+1);    }
  return res;
}
```

Output

```
5
F: m1=1, m2=1, n=5, i=2
F: &m1=134518656, &m2=134518660
F: &res=3221224516, &temp=3221224512
F: m1=2, m2=1, n=5, i=3
F: &m1=134518656, &m2=134518660
F: &res=3221224468, &temp=3221224464
F: m1=3, m2=2, n=5, i=4
F: &m1=134518656, &m2=134518660
F: &res=3221224420, &temp=3221224416
F: m1=5, m2=3, n=5, i=5
F: &m1=134518656, &m2=134518660
F: &res=3221224372, &temp=3221224368
F(5) = 8
```

Recursion vs. Iteration

- Repetition
  - Iteration: explicit loop
  - Recursion: repeated function calls
- Termination
  - Iteration: loop condition fails
  - Recursion: base case recognized
- Both can have infinite loops
- Balance
  - Choice between performance (iteration) and good software engineering (recursion).

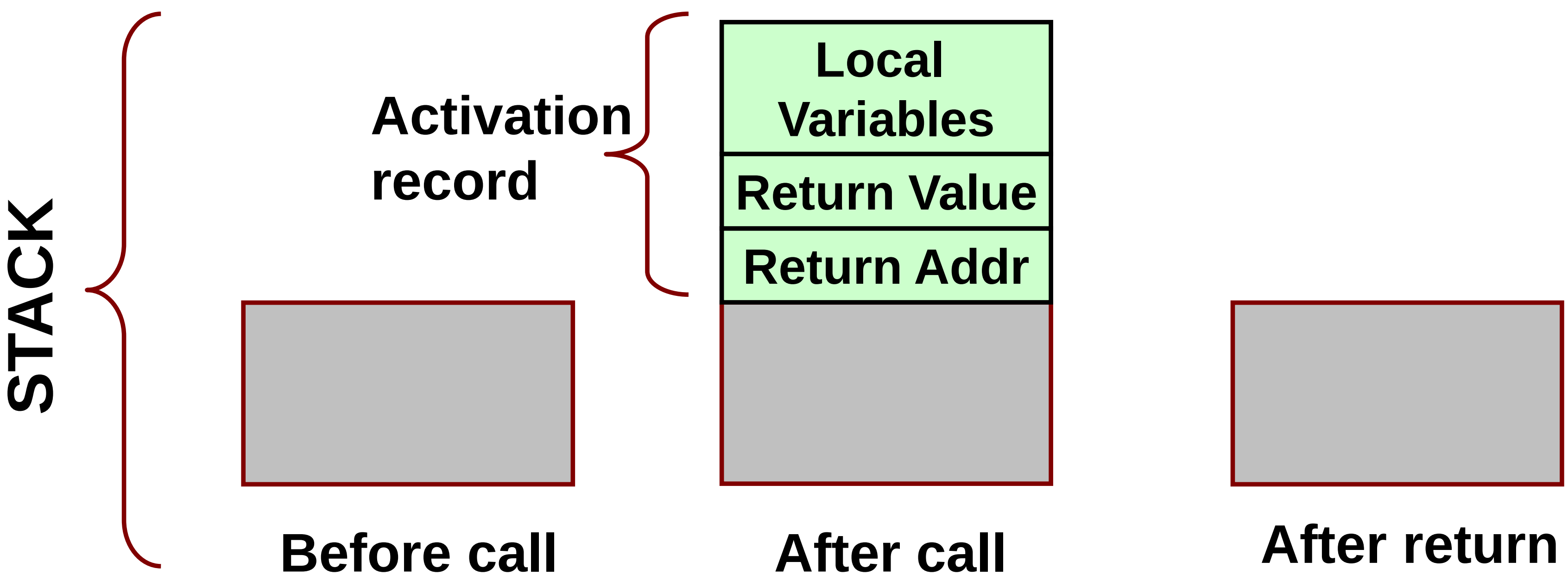
- Every recursive program can also be written without recursion
- Recursion is used for programming convenience, not for performance enhancement
- Sometimes, if the function being computed has a nice recurrence form, then a recursive code may be more readable

How are function calls implemented?

- The following applies in general, with minor variations that are implementation dependent
  - The system maintains a stack in memory
    - Stack is a last-in first-out structure
    - Two operations on stack, push and pop
  - Whenever there is a function call, the activation record gets pushed into the stack
    - Activation record consists of the return address in the calling program, the return value from the function, and the local variables inside the function

```
void main()
{
    ........
    x = gcd (a, b);
    ........
}
```

```
int gcd (int x, int y)
{
    ........
    ........
    return (result);
}
```

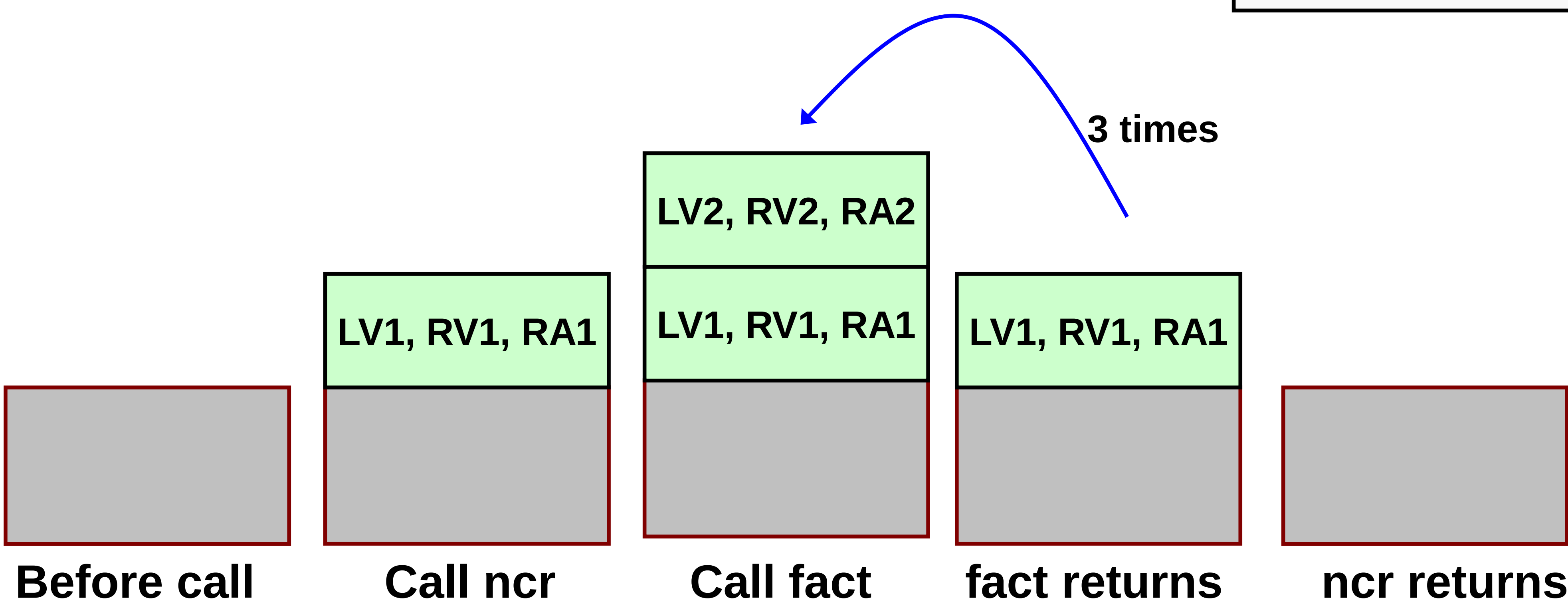


```
void main()
{
    ........
    x = ncr (a, b);
    ........
}
```

```
int ncr (int n, int r)
{
    return (fact(n)/
        fact(r)/fact(n-r));
}
```

3 times

```
int fact (int n)
{
    .........
    return (result);
}
```



What happens for recursive calls?

- What we have seen ….
  - Activation record gets pushed into the stack when a function call is made
  - Activation record is popped off the stack when the function returns
- In recursion, a function calls itself
  - Several function calls going on, with none of the function calls returning back
    - Activation records are pushed onto the stack continuously
    - Large stack space required

- Activation records keep popping off, when the termination condition of recursion is reached

- We shall illustrate the process by an example of computing factorial
  - Activation record looks like:

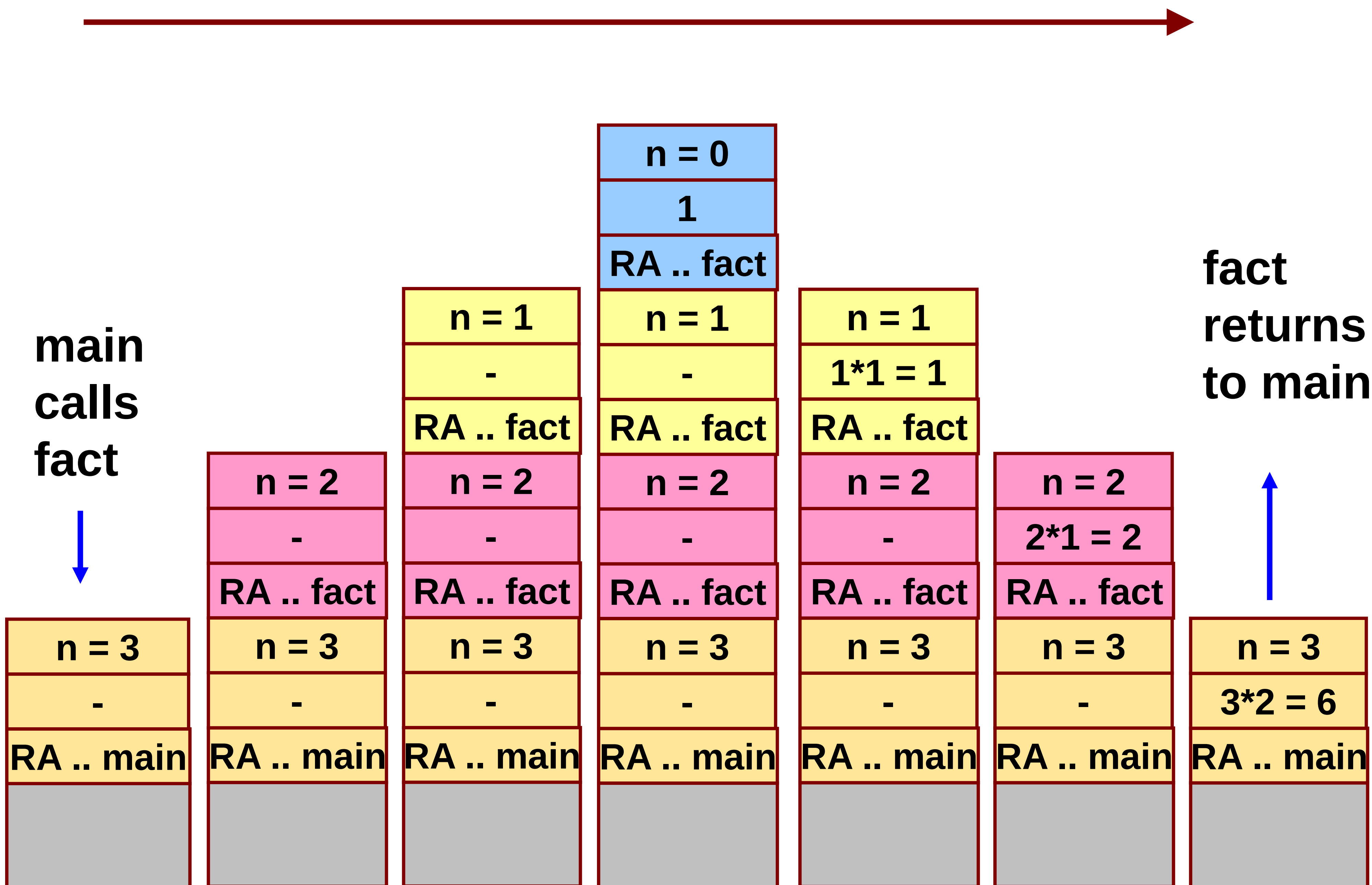
Local Variables
Return Value
Return Addr

Example:: main() calls fact(3)

```
void main()
{
   int  n;
   n = 3;
   printf (“%d \n”, fact(n) );
}
```

```
int  fact (n)
int  n;
{
    if   (n = = 0)
        return (1);
    else
        return  (n * fact(n-1));
}
```

TRACE OF THE STACK DURING EXECUTION



Do Yourself

- Trace the activation records for the following version of Fibonacci sequence

```
         int   f (int n)
         {
             int a, b;
             if  (n  < 2)   return (n);
             else  {
               a = f(n-1);
   X           b = f(n-2);
   Y           return (a+b);
             }
         }

         void main() {
             printf("Fib(4) is: %d \n", f(4));
main     }
```

Local Variables (n, a, b)
Return Value
Return Addr (either main, or X, or Y)