COL 7004 October 27, 29, 30

TUTORIAL SHEET 12

1. Suppose A is an $n \times n$ square matrix. Let $v_1, \ldots, v_r$ be eigenvectors of A corresponding to eigenvalues $\lambda_1, \ldots, \lambda_r$. Assume that the eigenvalues $\lambda_1, \ldots, \lambda_r$ are distinct. Show that $v_1, \ldots, v_r$ are linearly independent.

2. Suppose A is a symmetric matrix. Let v_1 and v_2 be eigenvectors corresponding to two different eigenvalues of A. By considering the term $v_1^T A v_2$, show that $\langle v_1, v_2 \rangle = 0$.

3. Write down the singular value decomposition of the following matrices:

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

4. Suppose you are given the singular value decomposition of an $m \times n$ matrix A, i.e., unitary matrices P and Q such that $Q^T AP$ is a diagonal matrix (a matrix is unitary if its columns are orthonormal). Show how to find a basis for the range and the nullspace of A.

5. Let G be an undirected graph with 100 vertices and 50 edges. What is the maximum and the minimum number of connected components it can have?

6. An undirected graph is said to be a tree if it does not have a cycle and it is connected.

   - Show that any tree must have a vertex of degree exactly 1.
   - Use induction to show that any tree with n vertices has $n-1$ edges.
   - Show that any undirected graph without any cycles and $n-1$ edges must be a tree (here n denotes the number of vertices in the graph).
   - Show that any connected undirected graph with $n-1$ edges must be a tree (here n denotes the number of vertices in the graph).