COL 351

TUTORIAL SHEET 10

1. Suppose you have algorithm A which given a graph G and a number k , outputs YES iff G has a vertex cover of size at most k. Assuming that A runs in polynomial time, show that you can find a vertex cover of minimum size in polynomial time.

   Solution: First, we can easily find the size of the minimum vertex cover – call it $k^\star$. Consider an edge u, v. Any solution of size k^* must pick either u or v. In other words, either $G - u$ or $G - v$ should have a vertex cover of size $k^* - 1$. Thus, here is a recursive algorithm $\mathcal{A}$ which given a graph H and a number k, either ouputs NO or outputs a vertex cover of size k. The algorithm $\mathcal{A}(H, k)$, where $k = 1$, simply checks if there is a single vertex in H which is a vertex cover. If so, it just returns this vertex, otherwise outputs NO. For $k > 1$: let (u, v) be an edge in H. Then we recursively run $\mathcal{A}(H - u, k - 1)$ and $\mathcal{A}(H - v, k - 1)$. If the answer is NO in both cases, we return NO. Otherwise say $\mathcal{A}(H - u, k - 1)$ returns a set S of size $k - 1$ which is a vertex cover of $H - u$. Then, $\mathcal{A}(H, k)$ returns $S \cup \{u\}$ (the other case for $H - v$ is similar). Finally, we run $\mathcal{A}$ on G, k^*.

2. The directed Hamiltonian Cycle Problem is as follows: given a directed graph G, is there a cycle which contains all the vertices ? Suppose you have a polynomial time algorithm for this problem. Show that you can also find such a cycle (if it exists) in polynomial time.

   Solution: Suppose G is Hamiltonian. Let C be any Hamiltonian cycle in G. Then, if we remove any edge not in C, the resulting graph will still be Hamiltonian. Thus, we get the following algorithm (let $\mathcal{A}$ denote the algorithm which given a graph, decides whether it is Hamiltonian or not): first run $\mathcal{A}$ on G to check if G is Hamiltonian or not. Assume G is Hamiltonian. While G has more than n edges, find an edge e in G such that $\mathcal{A}(G - e)$ returns true. As we argued above, there must exist such an edge – so we can try each edge in G and see if $\mathcal{A}(G - e)$ is true or not. Let e be such an edge. Then, we remove e, and repeat this process. Finally, when G has only n edges, these must form a Hamiltonian cycle.

3. The undirected Hamiltonian Cycle Problem can be defined similarly as above. The undirected Hamiltonian Path problem is as follows: given an undirected graph G, is there a path which contains all the vertices ? Show that the undirected Hamiltonian path is polynomial time reducible to the undirected Hamiltonian Cycle problem.

   Solution: Let $\mathcal{I}$ be an input to the Hamiltonian path problem. Note that $\mathcal{I}$ consists of an undirected graph G. We need to produce a graph G' such that G has a Hamiltonian path if and only if G' has a Hamiltonian cycle. We proceed as follows: add a new vertex v to the graph G and add edges between v and every vertex in G – call this graph G'. Now if P is a Hamiltonian path in G starting at vertex s and ending at t, then

v, s, P, t, v is a Hamiltonian cycle in G'. Conversely, if C is a Hamiltonian cycle in G', then removing the vertex v from C gives a Hamiltonian path in G.

4. Show that the undirected Hamiltonian cycle problem is reducible to the directed Hamiltonian cycle problem. Show that the directed Hamiltonian cycle problem is reducible to the undirected Hamiltonian cycle problem.

   Solution: We first reduce the undirected Hamiltonian cycle problem to the directed Hamiltonian cycle problem. Let G be an undirected graph, which is an input to the undirected Hamiltonian cycle problem. We need to produce a directed graph G' (in polynomial time) such that G has a Hamiltonian cycle if and only if G' has a (directed) Hamiltonian cycle. We construct G' by replacing each edge in G by two directed edges (going in opposite direction). It is easy to check that this reduction has the desired property.

   The reverse reduction is more tricky. Let $G = (V, E)$ be a directed graph, and from this we have to produce a graph $G' = (V', E')$ (in polynomial time) such that G has a Hamiltonian cycle if and only if G' has a Hamiltonian cycle. For every vertex $v \in G$, G' has three vertices - v', v'', v''' with edges $(v', v''), (v'', v''')$. For every directed edge (u, v) in G, we have the edge (u''', v') in G'. This completes the description of G'. Now suppose G has a Hamiltonian cycle: $v_1, v_2, \ldots, v_n$. Then $v_1', v_1'', v_1''', v_2', v_2'', v_2''', \ldots$ is a Hamiltonian cycle in G'. Now, suppose G' has a Hamiltonian cycle. Since each of the vertices v_i'' has degree 2, they must be preceded by v_i' and succeeded by v_i''' in this cycle (or the other way round). Therefore, if the vertices v_i'' appear in the cycle C in the order $v_1'', v_2'', \ldots, v_n''$, then it must be the case that the cycle looks like $v_1', v_1'', v_1''', v_2', v_2'', v_2''', v_3', \ldots$, and so $v_1, v_2, \ldots, v_n$, or $v_n, v_{n-1}, \ldots, v_1$ is a directed cycle in G.