Introduction to Vector Spaces

Why Vector Spaces?

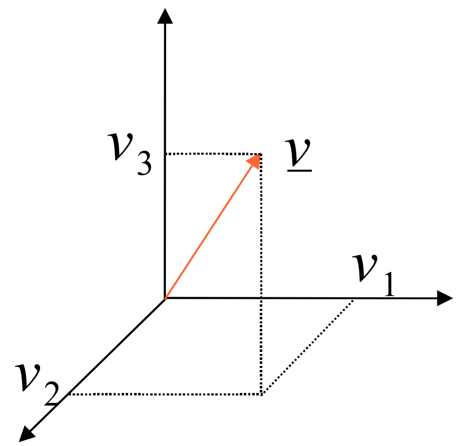
- Several issues better understood using vector spaces
  - point-to-point communications
  - error correction
  - multiple access

- Signals, codes etc. – vectors

Euclidean Space

- Vector $\underline{v}=(v_{1,}v_{2,}\ldots,v_n)$
- Addition $\underline{v}+\underline{w}=(v_1+w_{1,}v_2+w_{2,}\ldots,v_n+w_n)$
- Scalar Multiplication

$$a\,\underline{v}=(a\,v_1,a\,v_2,\ldots,a\,v_n)$$



- Basis vectors

$$\underline{e_1}=(1,0,\ldots,0)$$
$$\underline{e_2}=(0,1,\ldots,0)$$
$$\vdots$$
$$\underline{e_n}=(0,0,\ldots,1)$$

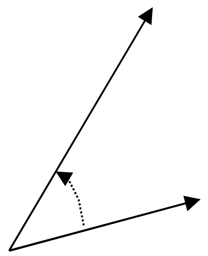
- Representation in-terms of basis

$$\underline{v}=\sum_{i=1}^{n}v_i\,\underline{e_i}$$

Euclidean Norm and Inner Product

- Inner Product of two vectors (related to angle)

$$\langle \underline{v}, \underline{w} \rangle = \sum_{i=1}^{n} v_i w_i = ||v|| \; ||w|| \; \cos(\underline{v} \sphericalangle \underline{w})$$



- Norm of vector

$$||v|| = \sqrt{\langle \underline{v}, \underline{v} \rangle} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

- Distance between vectors

$$d(\underline{v}, \underline{w}) = ||\underline{v} - \underline{w}|| = \sqrt{\sum_{i=1}^{n} (v_i - w_i)^2}$$

General Vector Space

- Set of vectors $\boldsymbol{V} = \{ \underline{v}, \underline{w}, ... \}$
- Addition results in a vector $\underline{v} + \underline{w} \in \boldsymbol{V}$
- Scalar Multiplication results in a vector

$$a\,\underline{v} \;\in\; \boldsymbol{V}\ \forall\, a \in \mathbb{R}$$

- Basis vectors - set of linearly independent vectors (no vector in the set can be written as a linear combination of others)

$\{ \underline{e_1}, \underline{e_2}, ... \} \subset V$ such that any vector can be written as linear combination of basis vectors

$$\underline{v} = \sum_i b_i \underline{e_i}$$

Normed and Inner Product Spaces

- Normed spaces are vector spaces with a norm $||\cdot||: V \to \mathbb{R}_+$
- Properties of norm
  - Positivity $||\underline{v}|| \geq 0 \quad \forall\, \underline{v} \in \boldsymbol{V}$
  - Scalability $||a\,\underline{v}|| = |a|\;||\underline{v}|| \quad \forall\, a \in \mathbb{R}$
  - Triangle inequality $||\underline{v}+\underline{w}|| \leq ||\underline{v}|| + ||\underline{w}||$

- Inner product spaces are vector spaces with inner product

$$\langle .\,,. \rangle : V \times V \to \mathbb{C}$$

- Properties of inner product
  - Conjugate Symmetry $\langle \underline{v}, \underline{w} \rangle = \overline{\langle \underline{w}, \underline{v} \rangle}$
  - Linearity $\langle a\,\underline{v}, \underline{w} \rangle = a \langle \underline{v}, \underline{w} \rangle \quad \forall\, a \in \mathbb{R}$
    $\langle \underline{v}+\underline{u}, \underline{w} \rangle = \langle \underline{v}, \underline{w} \rangle + \langle \underline{u}, \underline{w} \rangle \quad \forall\, \underline{v}, \underline{u}, \underline{w} \in \boldsymbol{V}$
  - non-negativity $\langle \underline{v}, \underline{v} \rangle \geq 0$

Space of 1-Dim Finite Energy Functions

- Call set of all 1-dim finite energy functions L^2
- Vector space $(denote\ function\ \ v(t)\ \ by\ \ \underline{v})$
  - Addition $Let\ \ \underline{h}=\underline{v}+\underline{w}\ \ then\ \ h(t)=v(t)+w(t)$
  - Scalar multiplication

$$Let\ \underline{h}=a\,\underline{v}\ \ then\ \ h(t)=a\,v(t)$$

- Inner Product

$$\langle \underline{v},\underline{w}\rangle=\int v(t)\overline{w(t)}dt$$

- Basis functions $\{e_f : f \in \mathbb{R}\}$

$$e_f=\exp\{2i\pi ft\}=\cos(2\pi ft)+i\sin(2\pi ft);\ \ i=\sqrt{-1}$$

- Can represent any function as linear combination of complex exponentials

Space of Binary Vectors

- N-dimensional vector

$$\underline{v}=(v_{1,}v_{2,.}..,v_n);\quad v_i\in\{0,1\}$$

- Addition: add corresponding bits modulo 2

$0 \oplus 0$	0
$0 \oplus 1$	1
$1 \oplus 0$	1
$1 \oplus 1$	0

- Scalars for multiplication are either 0 or 1
- Subtraction and addition are equivalent, why?
- Norm: total number of 1's in vector