Computational Geometry CSL 852

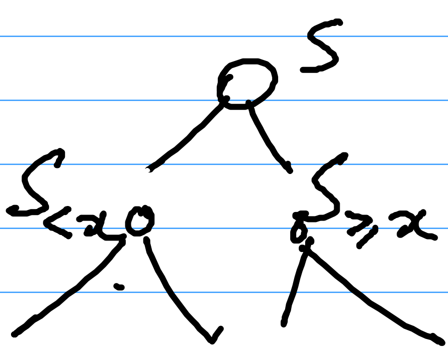
Lecture 20

Topic Randomized Incremental Construction

Quicksort : (1) pick a pivot element x at random

(2) Split the set S into two sets $S_{<x}$ $S_{\geq x}$

(3) Sort $S_{<x}$, $S_{\geq x}$ recursively



If $|S_{<x}| \sim |S_{\geq x}|$

then $\Rightarrow O(n \log n)$

Choice of x

$\Omega(n^2)$ pick the first element

uniformly at random — pick a random elem.

random choice

The expected running time of quicksort is $O(n \log n)$ where the expectation is over the choice of pivot elements.

pick the first element

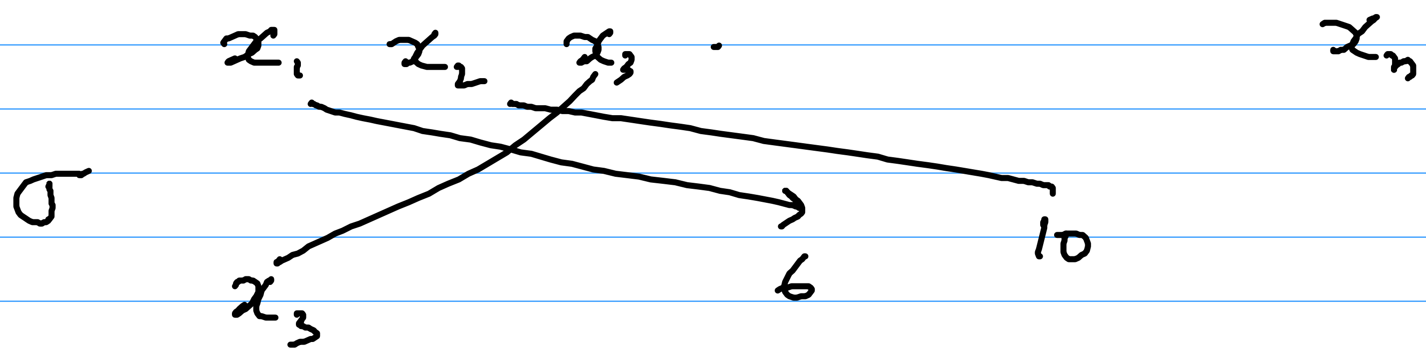
The average running time is $O(n \log n)$ where averaging is done over the $n!$ permutations

If we fix the permutation, $\Rightarrow$ the choices of the pivot elements are fixed

$T_{n,\sigma}$: is the running time of the input for a given permutation σ.

$$\overline{T}_n = \frac{1}{n!} \sum_{\substack{\sigma \text{: one} \\ \text{of the } n! \\ \text{perm}}} T_{n,\sigma}$$

- Generate a random permutation of the given set S of n elements : call it $\underline{\sigma}$



$\sigma^{-1}(1) : x_3$ $\qquad$ $\sigma^{-1}(6) : x_1$

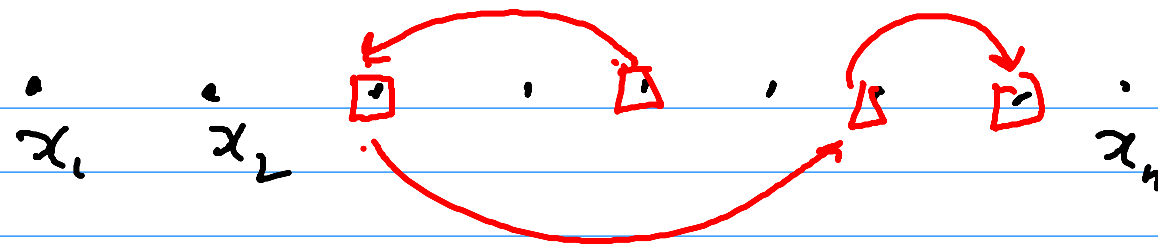
<u>Randomized Incremental Constr</u>

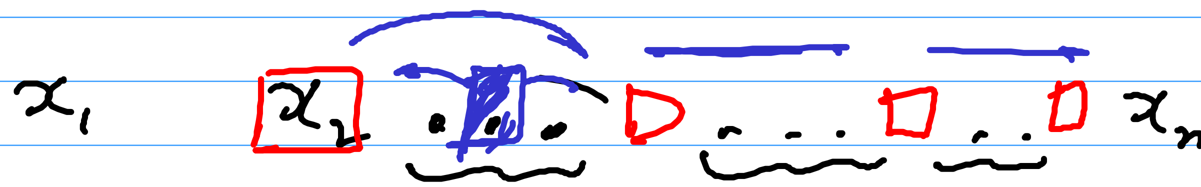
For $i = 1$ to n do

Consider the set of objects
$-S^i = \{\sigma^{-1}(1), \sigma^{-1}(2) \ldots \sigma^{-1}(i)\}$

- Compute the data structure / function on S^i (using information of S^{i-1})

- Report the final result corresponding to S^n





Pick an element at random of the given permutation

$\Updownarrow$

Pick the first element of the elements permuted randomly

$\overline{T}_n$: expected running time of the algorithm over the choice of the random permutation

$$= \frac{1}{n!} \sum_{\sigma \in \Pi} T_{n,\sigma}$$

Π : { set of $n!$ permut }

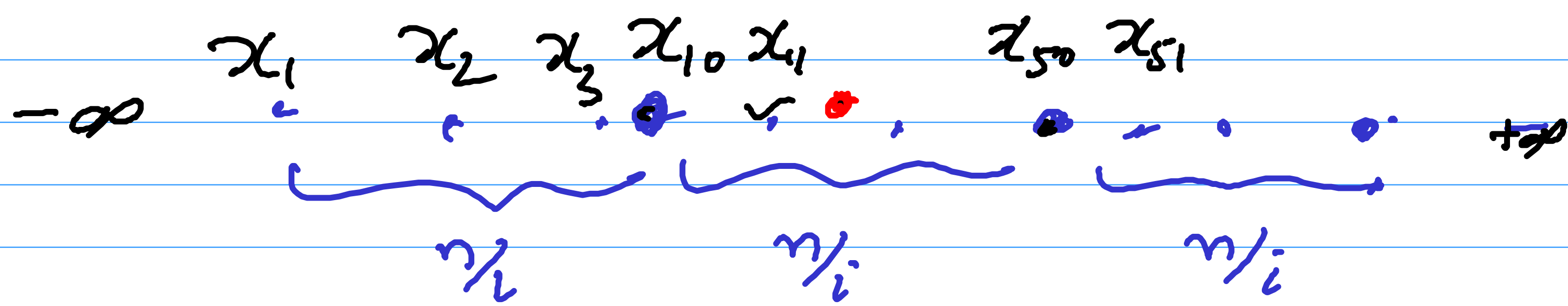
Examples :

1. Convex hulls

Maintain the convex hull of the first i points (of the random perm)

2. Closest pair :

Find the closest pair among the first i points



Work done for the i^{th} pivot

$$\sum_{i=1}^{n} O\left(\frac{n}{i}\right) = O\left(n \sum_{i=1}^{n} \frac{1}{i}\right) = O(n \log n)$$

$$\text{Prob}[Z_{ij} = 1] = \frac{2}{j} \qquad \sum_{j=1}^{n} \frac{2}{j} \le 2 \log n$$

$$Z_{i,j} = \begin{cases} 1 & \text{if } x_i \text{ is involved in the } j^{th} \text{ partitioning step} \\ 0 & \text{otherwise} \end{cases}$$

(indicator variable)

Total running time

$$T = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} Z_{i,j} \right)$$

total cost of a partitioning step

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} Z_{i,j}$$

For each element the total cost over all partitioning steps

$$E[T] = E\left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} Z_{i,j} \right)\right]$$

indicator r.v.

$$\leq n \cdot E\left(\sum_{j=1}^{n} Z_{i,j} \right)$$

$$E[X+Y] = E[X] + E[Y]$$

for <u>any</u> r.v. X, Y

$$\le n \cdot \sum_{j=1}^{n} \underbrace{E[Z_{ij}]}_{\Pr[x_i \text{ participates in the } j^{th} \text{ partition}]}$$

$$\le n \cdot \sum_{j=1}^{n} \underbrace{\Pr[Z_{ij} = 1]} = \frac{2}{j} \quad \le O(n \log n)$$

Fix the first j pivot elements

S^j $\quad$ $n-j$

# of permutation with S^j fixed

$$\Pr[Z_{ij} = 1 \mid S^j]$$

With S^j all permutation of j elements are equally likely

So any of the j elements can be the last element of the permutation

For a fixed element $y \in S^j$ the prob that y is the last element of a random permutation of $S^j = \frac{1}{j}$

So prob that $Z_{ij} = 1$ is the prob that one of the two bounding elements of x_i is the j^{th} pivot.

$$= \frac{2}{j}$$

Although this is conditioned on choice of S^j, the unconditional prob is same! Since all choices of S^j are equally likely