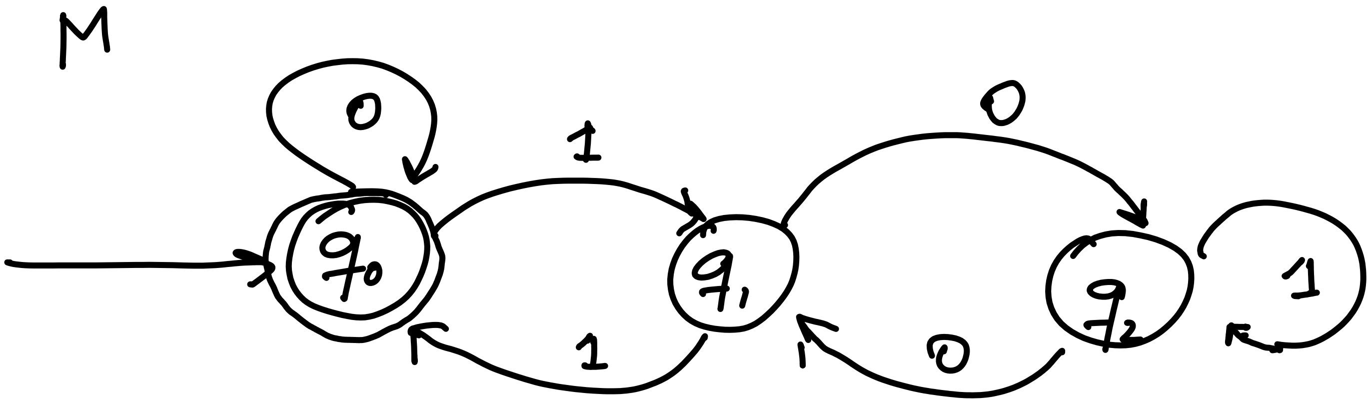
COL 352 Lecture 5 Jan 9, 2019

What does the following DFA accept



Difficult to characterize a language by looking at the DFA, but once we have a conjecture, we must prove it rigorously

For the above machine

$L(M) = \{0, 11, 00, 011, 110 \ldots\ldots\}$

Clearly $L(M)$ is infinite

Do we have a nice succint description

Claim : $L(M) = \{ w \mid bin(w) \text{ is divisible by } 3 \}$

$bin(w)$ is the value of a string w interpreted as a binary number

How do we prove it?

Induction on the length of the string

$\forall n \geq 1$, $w \in \Sigma^n$ is accepted by M iff $bin(w) \bmod 3 = 0$

Is the above easy to prove

Base cases : 0, 1 : strings of length one.

(i) 0 is accepted since q_0 is a final state

(ii) 1 is not accepted as q_1 is not a final state

I.H. Suppose the above is true for all strings of length n, then we must prove it for all strings of length $n+1$

Let $|w'| = n+1$ where

$$w' \begin{cases} \to w0 \\ \to w1 \end{cases} \qquad |w| = n$$

So w satisfies I.H.

Case. A : $w \in L(M)$ i.e. $bin(w) \bmod 3 = 0$

So $bin(w') = 2 \times bin(w) + 0/1$

$$\delta^*(q_0, w0) = \delta(\delta^*(q_0, w), 0) = \delta(q_0, 0) = q_0$$

i.e. $w.0$ is accepted and $bin(w0) \bmod 3 = 0$

So it is fine

$$\delta^*(q_0, w1) = \delta(\delta^*(q_0, w), 1) = \delta(q_0, 1) = q_1$$

$bin(W1) = [bin(w) \times 2 + 1] \bmod 3 = 1$

Since q_1 is not accepting it is fine

Case B : $w \notin L(M)$, i.e. $bin(w) \bmod 3 \neq 0$

What can we say about $w.0$ or $w.1$

From DFA, if $bin(W) \bmod 3 = 1$, then $w.1$ is accepted but if $bin(w) \bmod 3 = 2$ then $w.1$ is not accepted

So to complete the induction proof we need more information about w, i.e. $bin(w) \bmod 3 = 1$ or 2?

So the induction assertion must have more than just the property of state q_0 (which is the accepting state)

Attempt 2

Claim $\quad \forall n > 0, \ |w| = n$

(i) $\delta^*(q_0, w) = q_0$ iff $bin(w) \bmod 3 = 0$

(ii) $\delta^*(q_0, w) = q_1$ iff $bin(w) \bmod 3 = 1$

(iii) $\delta^*(q_0, w) = q_2$ iff $bin(w) \bmod 3 = 2$

[Or more compactly $\delta^*(q_0, w) = q_i$ iff $bin(w) \bmod 3 = i$

$i = 0, 1, 2$]

Proof **Base case** $|w| = 1$ i.e. $w = 0$ or 1

Check that $\delta(q_0, 0) = q_0$ $\delta(q_0, 1) = q_1$

Since no string of length 1 reaches q_2 it is vacuously true

I.H. Suppose the assertion is true for all strings w, $|w| = n$

$w' = w \cdot 0$ or $w \cdot 1$

Case A Case B

Case A $\text{bin}(w') = \text{bin}(w) \times 2$

$\text{bin}(w) \bmod 3 = 0$	$\text{bin}(w) \bmod 3 = 1$	$\text{bin}(w) \bmod 3 = 2$
$\Leftrightarrow$ $\delta^*(q_0, w) = q_0$ from I.H.	$\delta^*(q_0, w) = q_1$ from I.H.	$\delta^*(q_0, w) = q_2$ from I.H.
$\delta(q_0, 0) = q_0$ and $\text{bin}(w0) \bmod 3$ $= 2\,\text{bin}(w) \bmod 3 = 0$	$\delta(q_1, 0) = q_2$ and $\text{bin}(w0) \bmod 3$ $= 2\,\text{bin}(w) \bmod 3$ $= 2 \bmod 3$	$\delta(q_2, 0) = q_1$ and $\text{bin}(w0) \bmod 3$ $= 2 \cdot \text{bin}(w) \bmod 3$ $= 2 \times 2 \bmod 3 = 1$

Case B

Similarly you complete Case B

$w' = w1$

From these 2, we can conclude that

$\delta^*(q_0, w') = q_i$ iff $bin(w') \bmod 3 = i$

(Otherwise it would have shown up in one of the above cases)

The claim implies that

$L(M) = \{ w \mid bin(w) \bmod 3 = 0 \}$ since

q_0 is the only accepting state