Lecture 22, 23 COL 352 Feb 28, Mar 11

Pumping lemma for CFL

Let L be a CFL. Then there exists an integer n, s.t. for all $s \in L$ and $|s| \geq n$, s can be written as

$s = u \cdot v \cdot w \cdot x \cdot y$ where

$|v| + |x| \geq 1$ $\quad |v \cdot w\, x| \leq n$

s.t. $\forall i \geq 0$ $\quad u\, v^i\, w\, x^i\, y \in L$

Example: $L = \{a^i b^i c^i, \; i \geq 1\}$

Suppose L is a CFL $\Rightarrow$ P.L. can be applied

$u \quad \boxed{v \quad w \quad x} \quad y$

$$s = \underbrace{a \cdot a \cdots a}_{n} \; \underbrace{b \cdot b \cdots b}_{n} \; \underbrace{c\, c \cdots c}_{n}$$

Proof of PL for CFL

Assume that the CFL is in CNF

All prodn are $A \to BC \mid a$

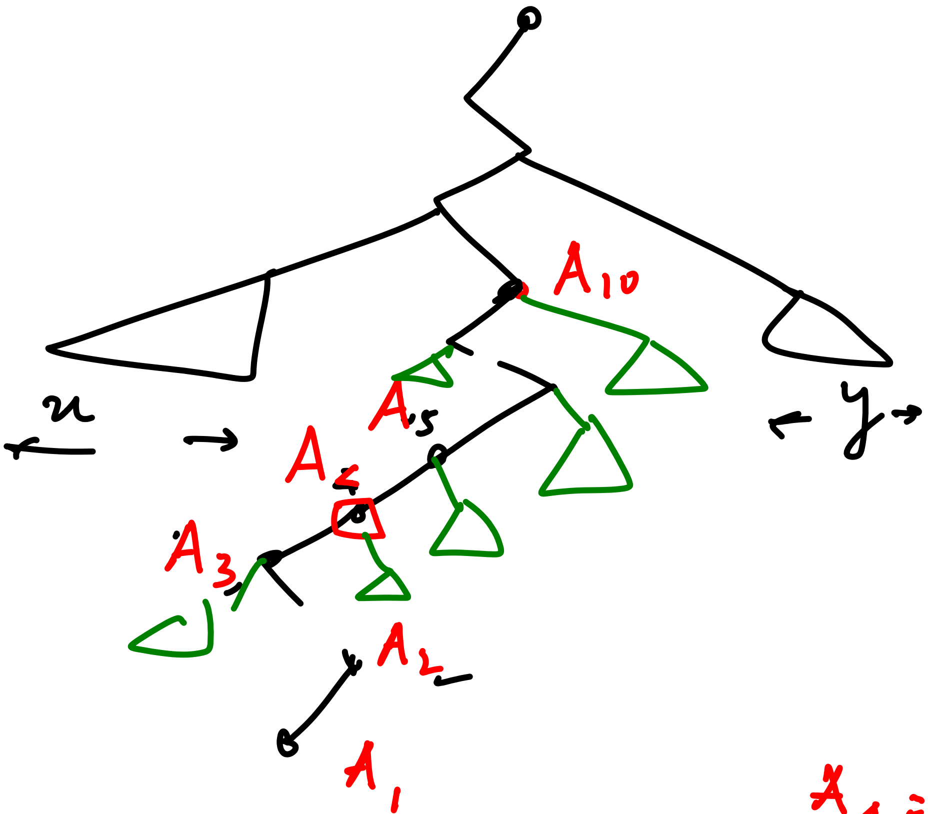
$s \in L$. Consider a derivation tree for s

$s_i \in \Sigma$

In any binary tree with 2^t leaf nodes there must be a path of length $\geq t$

$\Rightarrow$ If $t \geq |V|$ then some variable must be repeated in the derivation along the longest path

Note that a path of length t has $t+1$ vertices



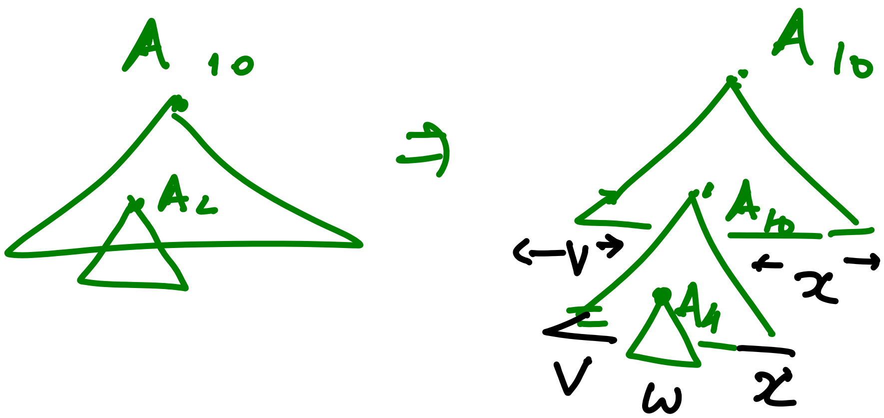
$|V| = k$

Path has length $\geq K$ and so # of variables is $\geq K+1$

$\Rightarrow$ Some variable is repeated

$A_i \in V$

$A_4 = A_{10}$



$n = 2^k$

P.L. n

$L = \{ ww \mid w \in (0+1)^* \}$

$L = \{ a^i b^j a^i b^j, \; i, j \geq 1 \}$

$aab\,aab \in L$ $\qquad$ $ab\,aab \notin L$

Is L CFL?

Consider $a^n b^n a^n b^n = u\,\boxed{\underline{v}\,w\,\underline{x}}\,y$ $\quad \xleftrightarrow{\;n\;}$

Exercise Show L is not CFL using PL

Is CFL closed under union?

L_1 is a CFL, say S_1 is the start symbol

L_2 is a CFL " S_2 is the start symb

Add the prodn $S \to S_1 \mid S_2$

Keep the variable names disjoint between L_1 and L_2

If L_1 is CFL and L_2 CFL

is $L_1 \cap L_2$ CFL ?

Can we do a product construction of two PDAs, M_1 M_2 ?

$Q = Q_1 \times Q_2$

Can we simulate 2 stacks by 1 stack?

$L_1 = \{ a^i b^i c^j \quad i,j \geq 1 \}$

$L_2 = \{ a^j b^i c^i , \quad i,j \geq 1 \}$

$L = L_1 \cap L_2 = \{ a^i b^i c^i , \quad i \geq 1 \}$

L is not a CFL, whereas L_1 and L_2 are CFL

CFLs are not closed under intersection

Corollary CFL are not closed under complementation

$$L_1 \cap L_2 = \overline{\bar{L}_1 \cup \bar{L}_2}$$

Special case : L_1 : CFL

L_2 : Regular

$L = L_1 \cap L_2$ is a CFL.

Use product construction : only one stack required.

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$$L = \{ ww \mid w \in (0+1)^* \}$$

Is L CFL ?

Suppose L is a CFL

Consider $L_1 = \{ a^i b^j a^i b^j \ , \ i,j \geq 1 \}$

$L_2 = a^+ b^+ a^+ b^+$

is regular

$$L \cap L_2 = L_1$$

Contradiction since L_1 is not CFL

Decision problems on CFL

1. Given a CFL is it

(i) finite (ii) infinite (iii) $\emptyset$

Let $K = |V|$, so $n = 2^K$ is the constant of the P.L. for a given CFG in Chomsky Normal Form

Claim : If $L \neq \phi$, then the shortest string in L, say s^* is such that $|s^*| < n$

Suppose not, then, let s' be the shortest string in L and $|s'| \geq n$. Then, from P.L. $s' = uvwxy$ such that $s'' = uv^0wx^0y \in L$ and $|s''| < s'$ since $|v| + |x| \geq 1$

Contradiction.

So it suffices to check if any string of length $\leq n-1$ is in L to check emptiness

Claim : If L is infinite then there must exist a string $s, \in L$ s.t. $n \leq |s| \leq 2n-1$

Suppose the shortest string in L has length $\geqslant 2n$ (Since L is infinite, it must have strings of length $\geqslant 2n$)

From P.L. $s = u\,v\,w\,x\,y$ s.t.

$$s_0 = u\,v^0\,w\,x^0\,y \in L$$

where $|s| - n \geqslant |s_0| < |s|$ since

$$n \geqslant |v| + |x| \geqslant 1$$

This implies that s is not the shortest string and this argument is valid for any $|s| \geqslant n$.

We want to show that there must exist a string in this range n —— $2n-1$

Suppose there is none, so apply the previous argument to s which is the shortest string $\in L$ whose length $\geqslant 2n$