A Logical Characterization of Efficiency Preorders

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Abstract. In this paper we present logical characterizations of two preorders, within the framework of Hennessy-Milner Logics. The two preorders (loosely termed bisimulation-based efficiency preorders) are on processes represented as labelled transition systems. The characterizations are particularly interesting as they explore preorders lying between strong and weak bisimilarity, guided by a principle of containment which is explained in the Introduction. Even though the proofs of the characterizations use standard methods, there are various subtleties introduced by the nature of the preorders and the logical operators needed to characterize them. The authors have not previously encountered the use of such operators in such simple logics.

Keywords Concurrency, transition systems, bisimulation, efficiency preorders, process efficiency, Hennessy-Milner Logic.

1 Introduction

In [5] a modal logic for reasoning about labelled transition systems was first defined which characterized the notions of simulations, strong and weak bisimulations. Subsequently the logic has been extended in various ways to include the modal μ-calculus and various behavioural equivalences and preordering relations on labelled transition systems. For a comprehensive account, the reader is referred to [12].

In all such formulations, a process is identified by the set of formulas of a logic that it satisfies. Given a behavioural equivalence relation on processes, a logical language L characterizes this equivalence relation precisely when two equivalent processes satisfy the same set of formulas of the logic.

Formally therefore, if $\mathbb{P}$ is a set of processes and L is a logic then we may identify each process $p \in \mathbb{P}$ with the set of formulas of L that it satisfies, i.e. $L(p) = \{\phi \in L \mid p \vDash \phi\}$ denotes the set of formulas of L that p satisfies. A behavioural equivalence $\cong$ on $\mathbb{P}$ is characterized by L whenever $p \cong q$ iff $L(p) = L(q)$. Similarly a behavioural preorder $\precsim$ on $\mathbb{P}$ is characterized by L whenever $p \precsim q$ iff $L(p) \subseteq L(q)$ or $p \precsim q$ iff $L(q) \subseteq L(p)$.

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Van Glabeek [14] has used precisely such formulations to characterize the various behavioural relations on concrete processes. In several instances, it has been shown that one behavioural relation R is coarser than another S, by showing that a less expressive logic characterizes R or that a more expressive logic characterizes S.

Such a formulation may actually be traced back to [5]. In it the authors present two modal logics - the stronger or more expressive one (subsequently referred to in the literature as *Hennessy-Milner Logic* or *HML*) characterizing strong bisimulation equivalence and the weaker or less expressive one (referred to by Stirling [12] as *Observable Hennessy-Milner Logic* or *OHML*) characterizing observational equivalence. The paper [5] also showed the characterization of trace equivalence and simulation equivalence in terms of sub-logics of HML, thus establishing that both trace equivalence and the simulation equivalence are coarser than bisimulation. The authors actually showed that for image-finite processes, HML with finitary conjunctions captured strong bisimilarity whereas HML without negation characterized *simulation equivalence*. *Trace equivalence* was characterized by removing both negation and conjunction from the logic.

In this paper we provide such logical characterizations for behavioural relations that lie strictly between strong and weak bisimilarity by defining modified *Hennessy-Milner Logics* for the purpose. We use the following properties as guiding principles in the design of the logics.

1. For any behavioural preorder $\leq$ and logical language L characterizing $\leq$, $p \leq q$ iff $L(p) \subseteq L(q)$. It then follows that the kernel of the preorder is an equivalence relation and is characterized by equality on sets of satisfying formulae. That is, $p \leq q$ and $q \leq p$ if and only if $L(p) = L(q)$.
2. Given preorders $\leq_1$, and $\leq_2$ characterized respectively by logics L_1 and L_2, $\leq_1 \subset \leq_2$ iff $L_1 \prec L_2$ where $L_1 \prec L_2$ denotes that L_1 is more expressive than L_2 (and hence can allow for finer distinctions to be made).

While adhering to these principles, it may be pointed out that so far the spectrum of behavioural relations that lie strictly between strong and weak bisimilarity has not really been explored in the literature. This includes some preorders defined by Milner and others ([11], [10], [3], [2]). In this paper we provide a characterization of the preorders defined in [3] and [2]. As we will show, the characterization of these preorders requires reasoning about linear orders within logics whose expressive power lies strictly between HML and OHML.

Efficiency-based preorders have been of interest to various people since they were first introduced in [1]. Several other authors have worked on obtaining similar preorders within the framework of extensionality (see [13], [6], [7], [8], [4]). Some of these works are based on extent theories such as testing and bisimulation in process algebra and Petri nets.

The paper is organized as follows. In the next section we review HML and OHML and show the characterizations of strong and weak bisimilarity respectively. Section 3 presents the logical characterization of the elaboration preorder [3] using an appropriately modified OHML. In this section we also highlight the principal differences

between the elaboration preorder and weak bisimilarity and hence the need for a more expressive logic than OHML to characterize elaboration. In section 4, we present the characterization of the efficiency preorder relation. Though the proofs in these sections are standard the operators introduced to characterize the two preorders are not standard. Section 5 is the conclusion and highlights further properties of the new modal operators that have been introduced to enrich OHML so as to characterize these preorders.

2 Modal Characterizations of Strong and Weak Bisimilarity

Here we review Hennessy-Milner Logic (HML) and Observable Hennessy-Milner Logic (OHML) and show how they characterize strong and weak bisimilarity respectively.

Let V be a set of *visible actions*, $\tau \notin V$ a distinguished *invisible action* and $Act = V \cup \{\tau\}$ the set of *actions*. A *labelled transition system (LTS)* is a 3-tuple $\langle \mathbb{P}, Act, \longrightarrow \rangle$, where $\mathbb{P}$ is a set of *process states* or *processes* and $\longrightarrow \subseteq \mathbb{P} \times Act \times \mathbb{P}$ is the *transition relation*. We use the notation $p \xrightarrow{a} q$ to denote $(p, a, q) \in \longrightarrow$ and refer to q as a *strong a-derivative* of p.

2.1 Strong Bisimulation

Definition 1. A *binary relation* $R \subseteq \mathbb{P} \times \mathbb{P}$ is a **strong simulation (SS)** if for every $\langle p, q \rangle \in R$ and $a \in Act$:

$$p \xrightarrow{a} p' \implies \exists q' : q \xrightarrow{a} q' \wedge p' R q'$$

It is a **strong bisimulation (SB)** if both R and R^{-1} are strong simulations. We write $p \sim q$ if there exists a strong bisimulation R such that pRq. The relation $\sim$ is called **strong bisimilarity**.

The relation $\sim$ is itself a strong bisimulation, and in fact, the largest one. In [9] the author gives a comprehensive account of the modal logic (HML) that characterizes strong bisimilarity. It is defined as follows.

Definition 2. The class L_{SB} of *strong bisimulation formulas* over Act is given by the following grammar (I is an indexing set, not necessarily finite).

$$\varphi ::= \ <a> \varphi \mid \bigwedge_{i \in I} \varphi_i \mid \neg \varphi$$

$tt \in L_{SB}$ where $tt \equiv \bigwedge_{i \in \emptyset} \varphi_i$. Similarly, $ff \in L_{SB}$ where $ff \equiv \neg tt$.

Definition 3. The *satisfaction relation* $\vDash \subseteq \mathbb{P} \times L_{SB}$ is defined recursively as follows:

– $p \vDash tt$ for all $p \in \mathbb{P}$.

- $p \vDash <a> \varphi$ for $a \in Act$ if $\exists p' \in \mathbb{P} : p \xrightarrow{a} p'$ and $p' \vDash \varphi$.
- $p \vDash \bigwedge_{i \in I} \varphi_i$ if $p \vDash \varphi_i$ for all $i \in I$.
- $p \vDash \neg \varphi$ if $p \nvDash \varphi$.

The set $SB(p)$ is defined as $SB(p) = \{\varphi \in L_{SB} \mid p \vDash \varphi\}$. We write $p \sqsubseteq_{SB} q$ iff $SB(p) \subseteq SB(q)$ and we write $p =_{SB} q$ iff $SB(p) = SB(q)$. The negation operator collapses the preorder $\sqsubseteq_{SB}$ to $=_{SB}$ as the following proposition shows.

Proposition 4 (Van Glabeek [14]). $p \sqsubseteq_{SB} q \Longleftrightarrow p =_{SB} q$.

Proof. If $\varphi \in SB(q) - SB(p)$ then $\neg \varphi \in SB(p) - SB(q)$. Hence, $p \sqsubseteq_{SB} q \Longleftrightarrow p =_{SB} q$. $\square$

Theorem 5 (Van Glabeek [14]). $p \sim q \Longleftrightarrow p =_{SB} q$.

Proof. ($\Longrightarrow$) By induction on the structure of φ. Since $p \sim q$, there exists a *strong bisimulation R* such that pRq.

- Let $p \vDash <a> \varphi$. Then there exists a $p' \in \mathbb{P}$ with $p \xrightarrow{a} p'$ and $p' \vDash \varphi$. Since pRq, there must be a $q' \in \mathbb{P}$ with $q \xrightarrow{a} q'$ and $p'Rq'$. So by induction $q' \vDash \varphi$ and hence $q \vDash <a> \varphi$.
  By symmetry, one also obtains $q \vDash <a> \varphi \Longrightarrow p \vDash <a> \varphi$.
- $p \vDash \bigwedge_{i \in I} \varphi_i \Longleftrightarrow \forall i \in I \, (p \vDash \varphi_i) \overset{induction}{\Longleftrightarrow} \forall i \in I \, (q \vDash \varphi_i) \Longleftrightarrow q \vDash \bigwedge_{i \in I} \varphi_i$.
- $p \vDash \neg \varphi \Longleftrightarrow p \nvDash \varphi \overset{induction}{\Longleftrightarrow} q \nvDash \varphi \Longleftrightarrow q \vDash \neg \varphi$.

($\Longleftarrow$) To prove that $p =_{SB} q \Longrightarrow p \sim q$, it suffices to establish that $=_{SB}$ is a strong bisimulation. If however, we show that $\sqsubseteq_{SB}$ is a strong simulation, then proposition 4 implies that $=_{SB} = \sqsubseteq_{SB} = \sqsubseteq_{SB}^{-1}$ is a strong bisimulation. We proceed to show that $\sqsubseteq_{SB}$ is a strong simulation.

Suppose, $p \sqsubseteq_{SB} q$ and $p \xrightarrow{a} p'$. Then $p \vDash <a> tt$ and $p \sqsubseteq_{SB} q$ implies q has at least one strong a-derivative. We have to show that $\exists q' \in \mathbb{P}$ with $q \xrightarrow{a} q'$ and $p' \sqsubseteq_{SB} q'$. Let

$$Q = \{q'' \in \mathbb{P} \mid q \xrightarrow{a} q'' \wedge p' \not\sqsubseteq_{SB} q''\}$$

For every $q'' \in Q$ there is a formula $\varphi_{q''} \in SB(p') - SB(q'')$. Now $<a> \bigwedge_{q'' \in Q} \varphi_{q''} \in SB(p) \subseteq SB(q)$. Therefore there must be a $q' \in \mathbb{P}$ with $q \xrightarrow{a} q'$ and $q' \vDash \bigwedge_{q'' \in Q} \varphi_{q''}$, which implies $q' \notin Q$.

$\square$

With negation in the logic, we may define the duals of the operators in L_{SB} as derived ones.

$$[a]\varphi \equiv \neg <a> \neg \varphi \qquad\qquad \bigvee_{i \in I} \varphi_i \equiv \neg \bigwedge_{i \in I} \neg \varphi_i$$

We may then define a negation-free language $L_{SB'}$.

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Definition 6. The class $L_{SB'}$ of *negation-free strong bisimulation formulas* over *Act* is given by the following grammar

$$\varphi ::= <a> \varphi \mid \bigwedge_{i \in I} \varphi_i \mid [a]\varphi \mid \bigvee_{i \in I} \varphi_i$$

with $tt \equiv \bigwedge_{i \in \emptyset} \varphi_i$ and $ff \equiv \bigvee_{i \in \emptyset} \varphi_i$.

Definition 7. The *satisfaction relation* $\vDash\, \subseteq \mathbb{P} \times L_{SB'}$ is defined recursively as in the case of L_{SB} with the clause for negation being replaced by the following clauses.

- $p \vDash ff$ for no $p \in \mathbb{P}$.
- $p \vDash [a]\varphi$ for $a \in Act$ if $\forall\, p' \in \mathbb{P} : p \xrightarrow{a} p' \implies p' \vDash \varphi$.
- $p \vDash \bigvee_{i \in I} \varphi_i$ if $p \vDash \varphi_i$ for some $i \in I$.

It is clear that $\neg\neg\varphi$ is equivalent to φ. It may be shown (refer [12]) that L_{SB} and $L_{SB'}$ are expressively equivalent and characterize the same equivalence.

2.2 Weak Bisimulation

We may define several other derived operators. Some of the relevant ones are given below.

$$<a>^0\, \varphi \equiv \varphi \qquad\qquad\qquad <a>^{m+1}\, \varphi \equiv <a><a>^m\, \varphi, m \in \mathbb{N}$$
$$\ll\gg \varphi \equiv \bigvee_{m \geq 0} <\tau>^m\, \varphi \qquad\qquad \ll a \gg \varphi \equiv \ll\gg<a>\ll\gg \varphi$$
$$[[\,]]\varphi \quad\;\; \equiv \neg \ll\gg \neg\varphi \qquad\qquad [[a]]\varphi \quad\;\; \equiv [[\,]][a][[\,]]\varphi$$

The last four operators correspond to the weak transition relation $\implies$ which is the smallest relation on processes such that

- $p \implies p$ for all processes p, and
- $p \xrightarrow{\tau} q$ and $q \implies r$ implies $p \implies r$

Further for any $a \in Act$,

- $p \xRightarrow{a} q$ if $p \implies \xrightarrow{a} \implies q$, and q is called a *weak a-derivative* or simply an *a-derivative* of p.
- $p \xRightarrow{\hat{a}} q$ denotes $p \implies q$ if $a = \tau$ and $p \xRightarrow{a} q$ otherwise.
- $\ll \hat{a} \gg$ denotes $\ll\gg$ if $a = \tau$ and $\ll a \gg$ otherwise.

Definition 8. A *binary relation $R \subseteq \mathbb{P} \times \mathbb{P}$* is a **weak simulation** if for every $\langle p,q \rangle \in R$ and $a \in Act$,

$$p \xrightarrow{a} p' \implies \exists q' : q \xRightarrow{\hat{a}} q' \wedge p'Rq'$$

R is a **weak bisimulation (WB)** if both R and R^{-1} are weak simulations. We write $p \approx q$ if there exists a weak bisimulation R such that pRq.

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Definition 9. The class L_{WB} of *weak bisimulation formulae* over *Act* is defined by the following BNF.

$$\varphi ::= \ <<\alpha>> \varphi \mid \bigwedge_{i \in I} \varphi_i \mid \neg\varphi$$

where $\alpha \in V$.

Note that in the light of our discussion on derived operators, L_{WB} is entirely contained in L_{SB}. It is also less expressive in the sense that it cannot express properties relating to the number of invisible moves a process might be able to make.

Definition 10. The *satisfaction relation* $\models \subseteq \mathbb{P} \times L_{WB}$ is defined recursively by:

- $p \models tt$ for all $p \in \mathbb{P}$.
- $p \models <<\alpha>> \varphi$ for $\alpha \in V$ if $\exists p' : p \overset{\alpha}{\Longrightarrow} p' \wedge p' \models \varphi$
- $p \models \bigwedge_{i \in I} \varphi_i$ if $p \models \varphi_i$ for all $i \in I$.
- $p \models \neg\varphi$ if $p \not\models \varphi$.

The set of all formulae that a process p satisfies is defined as $WB(p) = \{\varphi \in L_{WB} \mid p \models \varphi\}$. We write $p \sqsubseteq_{WB} q$ iff $WB(p) \subseteq WB(q)$ and we write $p =_{WB} q$ iff $WB(p) = WB(q)$. The proofs that $p \sqsubseteq_{WB} q$ implies $p =_{WB} q$ and that $p \approx q$ iff $p =_{WB} q$ (i.e. that L_{WB} characterizes weak bisimilarity) are similar to the corresponding proofs for strong bisimilarity and hence are omitted. It is also possible to define a negation-free logic $L_{WB'}$ (analogous to $L_{SB'}$) which characterizes weak bisimilarity.

3 Elaboration

In this section we give a logical characterization of the elaboration preorder defined in [3]. This logic builds upon L_{WB} and exhibits *controlled* use of counting.

Definition 11. A *binary relation* $R \subseteq \mathbb{P} \times \mathbb{P}$ is an **elaboration** iff for every $\langle p, q \rangle \in R$ the following conditions hold for every action $a \in Act$.

$$p \overset{a}{\longrightarrow} p' \Longrightarrow \exists q' : q \overset{\hat{a}}{\Longrightarrow} q' \wedge p' R q' \tag{1}$$

$$q \overset{a}{\longrightarrow} q' \Longrightarrow \exists p' : p \overset{a}{\Longrightarrow} p' \wedge p' R q' \tag{2}$$

We write $p \lesssim\!\!\!\lesssim q$ if there exists an elaboration R such that pRq.

Loosely speaking, if $p \lesssim\!\!\!\lesssim q$, then $p \approx q$ and for every execution sequence of p, there exists a possibly shorter execution sequence of q which exhibits the same visible behaviour. In [3] the authors illustrate this feature with a small example. They also show in the equational axiomatization of the corresponding precongruence (and congruence) for finite CCS processes, that it differs from that of observational congruence in just one equation. Whereas observational congruence satisfies the equation $a.\tau.p = a.p$, in the case of elaboration, $a.\tau.p \lesssim\!\!\!\lesssim a.p$ but the converse $a.p \lesssim\!\!\!\lesssim a.\tau.p$ does not always hold.

In that sense the precongruence is closer to observational congruence than any other relation defined so far in the literature.

It is easy to show from the above definition that every *elaboration* is a weak bisimulation. In particular, the only difference from that of weak bisimulation is in clause (2), where the $\overset{\hat{a}}{\Longrightarrow}$ is replaced by $\overset{a}{\Longrightarrow}$. We state without proof, a result (see [3]) that we use in the proof of lemma 19.

Lemma 12. *If $R \subseteq \mathbb{P} \times \mathbb{P}$ is an elaboration then for every $\langle p, q \rangle \in R$ the following conditions hold for every action $a \in Act$.*

$$p \overset{a}{\Longrightarrow} p' \implies \exists q' : q \overset{\hat{a}}{\Longrightarrow} q' \wedge p'Rq' \tag{3}$$

$$q \overset{a}{\Longrightarrow} q' \implies \exists p' : p \overset{a}{\Longrightarrow} p' \wedge p'Rq' \tag{4}$$

From lemma 12 it is clear that any HML-style characterization that is in accordance with the principles laid out in section 1 would enrich observable HML slightly in order to characterize the preorder. We define the following two-level grammar, whose lower level (defined by the meta-variable φ) is L_{WB}. The higher level (defined by the meta-variable π) expresses a certain weak form of counting. It introduces a new operator ε^k, (where k is a positive integer) but does not permit negation to precede any occurrence of ε^k. The operator $\ll \hat{a} \gg$ excludes $\ll \tau \gg$. Where counting of τ actions is not important, the lack of negation is compensated by the derived operators $[[\ldots]]$ and $\bigvee$.

Definition 13. The class L_E of *Elaboration formulae* over *Act* is given by the following two-level grammar, where $\alpha \in V$ and $a \in A$.

$$\varphi ::= \ll \alpha \gg \varphi \mid \bigwedge_{i \in I} \varphi_i \mid \neg \varphi$$

$$\pi ::= \varphi \mid \ll \hat{a} \gg \pi \mid \varepsilon^k \pi \mid [[a]]\pi \mid \bigwedge_{i \in I} \pi_i \mid \bigvee_{i \in I} \pi_i$$

Definition 14. The *satisfaction relation* $\vDash \subseteq \mathbb{P} \times L_E$ is defined recursively in a manner that should be obvious by now for all operators drawn from L_{WB}. So we restrict ourselves to the definitions of the operators $\ll \hat{a} \gg$, ε^k and $[[a]]$ respectively.

- $p \vDash \ll \hat{a} \gg \pi$ for $a \in Act$, if $\exists p' \in \mathbb{P} : p \overset{\hat{a}}{\Longrightarrow} p' \wedge p' \vDash \pi$.
- $p \vDash \varepsilon^k \pi$ for $k > 0$, if $\forall p' \in \mathbb{P} : p \overset{\tau^j}{\longrightarrow} p' \wedge p' \vDash \pi \implies j < k$
- $p \vDash [[a]]\pi$ for $a \in Act$ if for all $p' \in \mathbb{P} : p \overset{a}{\Longrightarrow} p' \implies p' \vDash \pi$.

$p \overset{\tau^j}{\longrightarrow} p'$ denotes that p may evolve to p' after performing j consecutive invisible actions. $E(p) = \{\pi \in L_E \mid p \vDash \pi\}$ and $E_{WB}(p) = \{\varphi \in L_{WB} \mid p \vDash \varphi\}$. We write $p \sqsubseteq_E q$ iff $E(p) \subseteq E(q)$ and $p =_E q$ iff $E(p) = E(q)$. Similarly, $p \sqsubseteq_{E_{WB}} q$ iff $E_{WB}(p) \subseteq E_{WB}(q)$ and $p =_{E_{WB}} q$ iff $E_{WB}(p) = E_{WB}(q)$.

To gain a deeper understanding, we give below some examples that illustrate the expressive power of L_E.

Example 15.

- The statement "*p is stable*" i.e. p cannot perform a τ action, may be expresed as $\varepsilon^1 tt$. More generally, $p \vDash \varepsilon^k tt$ if and only if p may perform *no more than* $(k-1)$ consecutive τ actions.
- The statement "*p converges*" i.e. p cannot perform an infinite sequence of τ actions, is expressed as $\bigvee_{k>0} \varepsilon^k tt$.
- The statement "*p diverges*" is however not expressible in the language since p diverges iff $p \nvDash \bigvee_{k>0} \varepsilon^k tt$.
- The statement $p \vDash \varepsilon^k ff$ means that "p can perform τ^m for all $m \geq k$". However, this statement does not necessarily imply that p diverges, unless p is also finitely branching.
- $\varepsilon^j \pi$ logically implies $\varepsilon^k \pi$ for all $0 < j < k$.
- For any $j > 0$, $\bigwedge_{k \geq j} \varepsilon^k \pi$ is logically equivalent to $\varepsilon^j \pi$.
- Statements such as "*p can do at least two consecutive τ actions*" are not expressible since there is no operator which can express lower bounds on the number of consecutive τ actions[1].

We now proceed to prove the characterization theorem. We begin with the following lemma which is clearly implied by the fact that every elaboration is a weak bisimulation and that $\sqsubseteq_{E_{WB}}$ and $=_{E_{WB}}$ coincide.

Lemma 16. $p \lesssim\kern-0.6em\raise0.3ex{\scriptscriptstyle\approx}\; q \implies p =_{E_{WB}} q$ *and hence* $p \lesssim\kern-0.6em\raise0.3ex{\scriptscriptstyle\approx}\; q \implies p \sqsubseteq_{E_{WB}} q$. $\qquad\qquad\square$

Theorem 17. The characterization. $p \lesssim\kern-0.6em\raise0.3ex{\scriptscriptstyle\approx}\; q$ *iff* $p \sqsubseteq_E q$. $\qquad\qquad\square$

We split the proof of theorem 17 into two parts.

Lemma 18. $p \sqsubseteq_E q$ *implies* $p \lesssim\kern-0.6em\raise0.3ex{\scriptscriptstyle\approx}\; q$.

Proof. We show that $\sqsubseteq_E$ is an elaboration. Let $E(p) \subseteq E(q)$. We need to prove both parts (1) and (2) of definition 11.

Part (1). Consider $p \xrightarrow{a} p'$ where $a \in Act$. Then $p \vDash \ll \hat{a} \gg \pi$ for any $\pi \in E(p')$ and $q \vDash \ll \hat{a} \gg \pi$ which implies $\exists q' : q \overset{\hat{a}}{\Longrightarrow} q' \wedge q' \vDash \pi$. We need to show that $\exists q' : q \overset{\hat{a}}{\Longrightarrow} q' \wedge E(p') \subseteq E(q')$.

Let
$$Q = \{ q'' \mid q \overset{\hat{a}}{\Longrightarrow} q'' \wedge E(p') \nsubseteq E(q'') \}$$

Then for each $q'' \in Q$, there exists a formula $\pi_{q''} \in E(p') - E(q'')$. This implies $p' \vDash \bigwedge_{q'' \in Q} \pi_{q''}$. Hence $p \vDash \pi$ where $\pi = \ll \hat{a} \gg \bigwedge_{q'' \in Q} \pi_{q''}$. That is, $\pi \in E(p) \subseteq E(q)$. Hence

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[1] To be able to express such statements, would require either the power of negation or the dual of ε^k. But allowing such operators would make the logic equivalent to L_{SB}, something we wish to avoid.

$\exists q' : q \overset{\hat{a}}{\Longrightarrow} q' \wedge q' \vDash \bigwedge_{q'' \in Q} \pi_{q''}$ and $q' \notin Q$, which shows that $E(p') \subseteq E(q')$, which needed to be proved.

Part (2). Suppose $q \overset{a}{\longrightarrow} q'$. We claim $\exists p' : p \overset{a}{\Longrightarrow} p'$. Suppose not. If $a \in V$ then for some $\varphi \in E_{WB}(q')$, $q \vDash \ll a \gg \varphi$ and $p \nvDash \ll a \gg \varphi$. This implies $p \vDash \neg \ll a \gg \varphi$ and $q \nvDash \neg \ll a \gg \varphi$ which contradicts $E_{WB}(p) \subseteq E_{WB}(q)$. On the other hand, if $a = \tau$, then $p \overset{\tau}{\not\Longrightarrow}$ implies $p \vDash \varepsilon^1 tt$. But since $q \overset{\tau}{\longrightarrow} q'$, $q \nvDash \varepsilon^1 tt$ which contradicts $E(p) \subseteq E(q)$.

Now, we need to show that $\exists p' : p \overset{a}{\Longrightarrow} p' \wedge E(p') \subseteq E(q')$. Suppose there is no such p'. Then $\forall p' : p \overset{a}{\Longrightarrow} p' \implies E(p') \nsubseteq E(q')$. Then for each weak a-derivative p'' of p, there exists a $\pi_{p''} \in E(p'') - E(q')$. Choose one such for each weak a-derivative p'' of p and collect them in a set Π. Then we have $p \vDash [[a]](\bigvee \Pi)$, but $q \nvDash [[a]](\bigvee \Pi)$, which contradicts $E(p) \subseteq E(q)$. Hence $\exists p' : p \overset{a}{\Longrightarrow} p' \wedge E(p') \subseteq E(q')$. □

Lemma 19. $p \underset{\approx}{\lesssim} q$ implies $p \sqsubseteq_E q$ i.e. $E(p) \subseteq E(q)$.

Proof. We need to show $p \underset{\approx}{\lesssim} q \implies (p \vDash \pi \implies q \vDash \pi)$. We prove this by induction on the structure of π

- $p \vDash \varphi$. Then $q \vDash \varphi$ follows from lemma 16.
- $p \vDash \ll \hat{a} \gg \pi$, $a \in Act$. Then $\exists p' : p \overset{\hat{a}}{\Longrightarrow} p' \wedge p' \vDash \pi$ and since $p \underset{\approx}{\lesssim} q$, $\exists q' : q \overset{\hat{a}}{\Longrightarrow} q' \wedge p' \underset{\approx}{\lesssim} q'$. Therefore, by induction hypothesis, $q' \vDash \pi \implies q \vDash \ll \hat{a} \gg \pi$.
- $p \vDash \varepsilon^k \pi$. Then $\forall p' : p \overset{\tau^j}{\longrightarrow} p' \wedge p' \vDash \pi \implies j < k$. Since $p \underset{\approx}{\lesssim} q$, we have, for every such p' and j there exist q' and m respectively, such that $q \overset{\tau^m}{\longrightarrow} q' \wedge p' \underset{\approx}{\lesssim} q'$. Therefore by the induction hypothesis, $q' \vDash \pi$. We claim that, $m \geq k$ is impossible. Suppose not; then $q \overset{\tau^m}{\longrightarrow} q' \wedge p' \underset{\approx}{\lesssim} q'$ where $m \geq k$, for some q' and m. Then $\exists p'' : p \overset{\tau^j}{\longrightarrow} p'' \wedge p'' \underset{\approx}{\lesssim} q'$ for some $j \geq m \geq k$. But then $p \nvDash \varepsilon^k \pi$ which is a contradiction. Hence $q \vDash \varepsilon^k \pi$.
- $p \vDash [[a]]\pi$. Then $\forall p' : p \overset{a}{\Longrightarrow} p' \implies p' \vDash \pi$. Since $p \underset{\approx}{\lesssim} q$, we have, for every such p', $\exists q' : q \overset{\hat{a}}{\Longrightarrow} q' \wedge p' \underset{\approx}{\lesssim} q'$. By the induction hypothesis $E(p') \subseteq E(q')$ and $q' \vDash \pi$. We now claim that there does not exist any q' such that $q \overset{a}{\Longrightarrow} q'$ and $q' \nvDash \pi$. Suppose the claim is false. Then we have $\exists q' : q \overset{a}{\Longrightarrow} q' \wedge q' \nvDash \pi$. This implies (by lemma 12) $\exists p' : p \overset{a}{\Longrightarrow} p' \wedge p' \underset{\approx}{\lesssim} q' \nvDash \pi$. Again by induction hypothesis $E(p') \subseteq E(q')$ and hence $p' \nvDash \pi$. But this contradicts the assumption $p \vDash [[a]]\pi$. Hence the claim is false.
- $p \vDash \bigwedge_{i \in I} \pi_i$. Then $p \vDash \pi_i$ for all $i \in I$ and by the induction hypothesis, $q \vDash \pi_i$ for all $i \in I$ which implies $q \vDash \bigwedge_{i \in I} \pi_i$
- $p \vDash \bigvee_{i \in I} \pi_i$. Then $p \vDash \pi_i$ for at least one $i \in I$ and by induction hypothesis it follows that $q \vDash \bigvee_{i \in I} \pi_i$. □

The above proof has been presented in detail to highlight the steps for the operators at the higher level, especially the operator ε^k. Note that the proof would not go through if we allowed $\ll \tau \gg$ in the logic.

4 Efficiency Prebisimulation

As we did for *elaboration* in the previous section, we now present a logical characterization for *efficiency prebisimulation* [2] using a logic which is more expressive than L_E but not as expressive as L_{SB}.

The following formulation [2] gives a simple definition of the efficiency preorder. Closely related preorders have been defined independently by Milner in [10] and [11], though neither their algebraic nor logical characterizations have been presented.

Definition 20. A binary relation $R \subseteq P \times P$ is an **efficiency prebisimulation (EP)** iff for every $\langle p, q \rangle \in R$, $\alpha \in V$, $a \in Act$ the following conditions are satisfied.

$$p \overset{\alpha}{\longrightarrow} p' \Longrightarrow \exists q' : q \overset{\alpha}{\longrightarrow} q' \wedge p'Rq' \tag{5}$$

$$p \overset{\tau}{\longrightarrow} p' \Longrightarrow p'Rq \vee (\exists q' : q \overset{\tau}{\longrightarrow} q' \wedge p'Rq') \tag{6}$$

$$q \overset{a}{\longrightarrow} q' \Longrightarrow \exists p' : p \overset{a}{\Longrightarrow} p' \wedge p'Rq' \tag{7}$$

We write $p \lesssim q$ if there exists an efficiency prebisimulation R such that pRq.

Intuitively, $p \lesssim q$ means that for every execution sequence that p may perform, it is possible to find a possibly shorter sequence that q may perform with the same visible content, and conversely, for any sequence that q may perform it is possible to find a possibly longer sequence that p may perform with the same visible content. In general, both the preorders $\lesssim$ and $\lessapprox$ represent comparisons between observationally equivalent processes and order them differently on the amount of internal computation they may perform. We refer to both preorders as efficiency-based preorders, but $\lesssim$ having been christened "efficiency preorder" earlier we continue to refer to it by the same name. We refer the reader to [2] and [1] for intuitively appealing examples and for the axiomatization of finite CCS processes.

In [3] it has been shown that $\sim \subset \lesssim \subset \lessapprox \subset \approx$ (where all the containments are strict). This suggests that we require a logic that is more expressive than L_E, but less than L_{SB}. This logic is very similar in structure to the logic for *elaboration* and is again defined in two levels with the lower level of weak formulae being the same.

Definition 21. The class L_{EP} of *efficiency prebisimulation formulae* over *Act* is given by the following grammar. As before, we use α to denote a visible action and a to denote any action.

$$\varphi ::= \; \ll\!\alpha\!\gg \varphi \mid \bigwedge_{i \in I} \varphi_i \mid \neg\varphi$$

$$\pi ::= \; \varphi \mid <\!\alpha\!> \pi \mid (\tau)\pi \mid \varepsilon^k\pi \mid [[a]]\pi \mid \bigwedge_{i \in I} \pi_i \mid \bigvee_{i \in I} \pi_i$$

Note that the index set I may be infinite. Also negation ($\neg$) as in the case of L_E, is available only for *weak formulae*. The language L_{EP} is the set of all formulae π and the language L_{WB} is the set of all weak formulae φ. Note that $L_{WB} \subseteq L_{EP}$.

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Definition 22. The *satisfaction relation* $\vDash\, \subseteq\, \mathbb{P} \times L_{EP}$ is defined recursively. As before, we omit those clauses which have been previously presented and restrict ourselves to defining the clauses for the new operators.

- $p \vDash\, <\alpha> \pi$ for $\alpha \in V$ if for some $p' \in \mathbb{P} : p \xrightarrow{\alpha} p'$ and $p' \vDash \pi$
- $p \vDash (\tau)\pi$ if $p \vDash \pi$ or for some $p' \in \mathbb{P} : p \xrightarrow{\tau} p'$ and $p' \vDash \pi$

$EP(p) = \{\pi \in L_{EP} \mid p \vDash \pi\}$ and $EP_{WB}(p) = \{\varphi \in L_{WB} \mid p \vDash \varphi\}$. Further, $p \sqsubseteq_{EP} q$ iff $EP(p) \subseteq EP(q)$, and $p =_{EP} q$ iff $EP(p) = EP(q)$. As before, $p \sqsubseteq_{EP_{WB}} q$ iff $EP_{WB}(p) \subseteq EP_{WB}(q)$ and $p =_{EP_{WB}} q$ iff $EP_{WB}(p) = EP_{WB}(q)$.

As for the expressiveness of this language, notice that

- we allow the "strong possibility" modality ("$<\alpha>$") from L_{SB}, but neither negation nor "strong necessity" ("$[\alpha]$").
- The operator "$<\tau>$" is replaced by the weaker prefix operator "(τ)" .
- The operator ε^k in the formula $\varepsilon^k\pi$ excludes the possibility of a process being able to perform more than $k-1$ initial τ actions and reaching a state satisfying π.
- On the other hand, p satisfying the formula $(\tau)^k\pi$, $k > 0$ (obtained by prefixing π by k occurrences of (τ)) asserts the existence of a τ^j derivative of p (for some j, $0 \leq j \leq k$) which satisfies π.

There are no formulae in L_{EP} equivalent to the HML formula $\neg <\tau><\tau> \phi$. Even a statement such as "*p has a strong τ^2-derivative that satisfies π*" can only be inferred if it is known that $p \nvDash \pi$, $p \nvDash (\tau)\pi$ and $p \vDash (\tau)^2\pi$.

We now proceed to give a proof outline of the characterization theorem. We begin with the following lemma which is clearly implied by the fact that every efficiency prebisimulation is a weak bisimulation and that $\sqsubseteq_{EP_{WB}}$ and $=_{EP_{WB}}$ coincide.

Lemma 23. $p \lesssim q \implies p =_{EP_{WB}} q$ *and hence* $p \lesssim q \implies p \sqsubseteq_{EP_{WB}} q$. $\qquad\square$

Theorem 24. The characterization. $p \lesssim q$ *iff* $p \sqsubseteq_{EP} q$. $\qquad\square$.

We prove this theorem in two parts. We omit most of the routine details and concentrate on only the operators that have been newly introduced.

Lemma 25. $p \sqsubseteq_{EP} q$ *implies* $p \lesssim q$.

Proof. We show that $\sqsubseteq_{EP}$ is an efficiency prebisimulation. We prove the various parts viz. (5), (6) and (7) of definition (20), assuming $EP(p) \subseteq EP(q)$.

<u>Part (5).</u> Suppose $p \xrightarrow{\alpha} p'$, $\alpha \in V$. Then $p \vDash\, <\alpha> \pi$ for each $\pi \in EP(p')$. Clearly $q \vDash\, <\alpha> \pi$ and there exists q' such that $q \xrightarrow{\alpha} q'$ and $q' \vDash \pi$. We need to show $\exists q' :$

$q \xrightarrow{\alpha} q' \wedge EP(p') \subseteq EP(q')$. We use an argument that has now become routine viz. consider the set $Q = \{q'' | q \xrightarrow{\alpha} q'' \wedge EP(p') \not\subseteq EP(q'')\}$ and proceed as before.

Part (6). Suppose $p \xrightarrow{\tau} p'$, then for each $\pi \in EP(p')$ we have $p \vDash (\tau)\pi$ and $q \vDash (\tau)\pi$. Then if $q \nvDash \pi$ there must exist a q' such that $q \xrightarrow{\tau} q'$ and $q' \vDash \pi$. It then suffices to show that $\exists q' : q \xrightarrow{\tau} q' \wedge EP(p') \subseteq EP(q')$. We may show this in a manner similar to that of Part 5.

Part (7). The proof of this part is similar to Part (2) of the proof of lemma 18. $\square$

Lemma 26. $p \lesssim q$ implies $p \sqsubseteq_{EP} q$.

Proof. Assume $p \lesssim q$. We again use induction on the structure of formulae. For any formula ϕ from EP_{WB}, lemma 23 assures us that $p \vDash \phi$ implies $q \vDash \phi$. Of the rest of the cases from the language $L_{EP} - L_{EP_{WB}}$ we consider only the following:

- $p \vDash <\alpha> \pi, \alpha \in V$. Then for some p', we have $p \xrightarrow{\alpha} p'$ and $p' \vDash \pi$. Hence $\exists q' : q \xrightarrow{\alpha} q' \wedge p' \lesssim q'$. By the induction hypothesis, $EP(p') \subseteq EP(q')$ and so we get $q \vDash <\alpha> \pi$.
- $p \vDash (\tau)\pi$. If $p \vDash \pi$ then by the induction hypothesis $q \vDash \pi$ and so it follows that $q \vDash (\tau)\pi$. On the other hand, if $p \nvDash \pi$, then $\exists p' : p \xrightarrow{\tau} p' \wedge p' \vDash \pi$. From $p \lesssim q$ we have, either $p' \lesssim q$ or $\exists q' : q \xrightarrow{\tau} q' \wedge p' \lesssim q'$. If $p' \lesssim q$ then $p' \vDash \pi$ implies $q \vDash \pi$ by the induction hypothesis. If $p' \not\lesssim q$ then $p' \vDash \pi$ implies $q' \vDash \pi$ by the induction hypothesis and it follows that $q \vDash (\tau)\pi$.
- $p \vDash \varepsilon^k \pi, k > 0$. This is again proved in a manner analogous to the corresponding proof for elaboration. $\square$

5 Conclusion

In the foregoing sections we have introduced two new operators into Hennessy-Milner Logic (HML) viz. ε^k and (τ). We have shown and characterized preorders lying strictly between strong and weak bisimilarity using versions of HML and OHML with these new operators.

At the outset, we would first of all like to be convinced that their introduction into L_{SB} does not in any way alter the expressive power of HML. It is easy to see from their semantics that these operators enjoy the following equivalences. For any formula π in L_E or L_{EP}, let $\tilde{\pi}$ denote the equivalent formula in L_{SB}. We then have

$$(\tau)\pi \equiv \tilde{\pi} \vee <\tau> \tilde{\pi} \tag{8}$$

$$\varepsilon^k \pi \equiv \bigwedge_{j \geq k} [\tau]^j \neg \tilde{\pi} \tag{9}$$

While (8) is obvious from its semantics, it is fairly easy to derive (9) by noting that $p \nvDash \varepsilon^k \pi$ precisely when $\exists p' : p \xrightarrow{\tau^j} p' \wedge j \geq k \wedge (p' \vDash \pi)$.

These operators also satisfy some other properties. For example, any formula $\pi \in L_{EP}$ logically implies $(\tau)\pi$. And for any $k > 0$ we have $(\tau)^k \pi \equiv \bigvee_{j \leq k} (\tau)^j \pi$. Further, $(\tau)^k \pi \implies (\tau)^{k+1} \pi$. Similarly, in both L_E and L_{EP}, for any $k > 0$, we have $\varepsilon^k \pi \implies \varepsilon^{k+1} \pi$. For any set S of positive integers, let $inf(S)$ and $sup(S)$ denote the minimum and maximum (provided it exists) elements of S respectively. We then have the following identities.

$$\bigwedge_{j \in S} (\tau)^j \pi \equiv (\tau)^{inf(S)} \pi \ , \qquad\qquad \bigwedge_{j \in S} \varepsilon^j \pi \equiv \varepsilon^{inf(S)} \pi \qquad\qquad \text{and}$$
$$\bigvee_{j \in S} (\tau)^j \pi \equiv (\tau)^{sup(S)} \pi \ , \qquad\qquad \bigvee_{j \in S} \varepsilon^j \pi \equiv \varepsilon^{sup(S)} \pi \ \ \text{if } S \text{ is finite.}$$

These properties enable us to follow the guiding principle that a preorder should be characterized by containment on sets of formulae.

If we were to extend the language to allow ε^0 as an operator, we would have that $p \vDash \varepsilon^0 \pi$ implies $p \nvDash \pi$. Then we would also be able to express the statement "*p diverges*" by the formula $p \vDash \varepsilon^0 \bigvee_{k > 0} \varepsilon^k tt$ and the statement "*p can perform at least two τ actions*" by $\varepsilon^0 \varepsilon^2 tt$.

The results that we have presented in this paper offer interesting technical insights into the construction of logical characterizations of preorders guided by the principles enunciated in section 1. As far as we are aware, in the literature, there are no logical characterizations of behavioural relations that lie strictly between strong and weak bisimilarity.

In future work, we hope to study and understand more about these operators and explore their interactions with fixpoint operators within a modal μ-calculus setting.

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