Abstract

In this thesis we present a formulation of the conformance relation for the fusion calculus. As described in [?], conformace is a precongruence relation for concurrent systems. We extend the notion of conformance as a refinement relation to the Fusion Calculus[?]. A complete axiomatisation for conformance is given and we end with some examples of modelling in Fusion calculus.

Certificate

This is to certify that this dissertation titled “Conformance Relation for Fusion Calculus”, and submitted by Vishal Grover(98162) and Ashin P. Thariyan(98129) towards the fulfillment of the requirements for the award of the degree of Bachelor of Technology in Computer Science and Engineering from the Indian Institute of Technology, Delhi is a correct representation of the task undertaken and work completed by them under my guidance and supervision. To the best of my knowledge, this work is entirely original and has not been submitted elsewhere for the award of any degree or diploma.

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Contents

List of Figures

List of Tables

Chapter 1

fintroduction

With the increasing and widespread use of computing systems, it has become imperative to ensure the correctness of a system prior to its implementation. The need to reason with exactness and trust has led to development of mathematical tools for the purpose. For instance, complexity theory is applied to make cryptographic systems safer. The λ-calculus and the concept of types is an important ingredient in the present day programming languages. The mathematical tools provide a valuable medium for a study of computing systems. In this thesis we study concurrent systems with the help of process calculi.

1.1 Background

Process calculi are tool to study concurrent systems, where concurrnet processes interact as a part of a large system. The interaction between components of a concurrent system can be described in different ways. For instance, we can model the components as communicating through variables which are shared by all of them. Alternatively, we can model the sytem with an explicit view of the communication, that is two components interact by passing messages. The process calculus has its roots in the Calculus Of Communicating Systems of Milner [?]. The fusion calculus[?], a direct descendant of the π-calculus [?], brings the two major views of communication - shared variables and message passing - together in a single framework. It is a calculus for mobile systems, where the links between the interacting processes are dynamic.

An important part of reasoning about interactive agents is to decide when do two processes have an equivalent behaviour, with respect to its interaction with external agents. For this purpose bisimulation is used. Another important aspect is to compare two equivalent processes. This is where a preorder relation is used.

An efficiency preorder is a reflexive and transitive relation over the process description language, which also takes into consideration behavioural equivalence. Efficiency preoders were originally motivated and defined in [**?**] and [**?**] as being relations which could compare equivalent processes on an abstract measure of computational complexity. However, it is also possible to regards these preorders as defining refinement relations which preserve behavioural equivalence. Hence, if p and q are two agents, and if $\leq$ is such a relation then p $\leq$ q means : p and q are behaviourally equivalent and p is a refinement of q. In this thesis, we present such a relation for the fusion calculus.

1.2 Overview

In the next chapter we present a brief introduction to the fusion calculus. We define the syntax of the calculus, followed by a presentation of the operational semantics by a labelled transition system. Finally we present the bisimulation relations.

In chapter 3, we define elaboration and we prove that it is a preorder on the agents of the fusion calculus. Then we motivate and define the conformance relation. We prove that it is a preorder preserved over all the operators of the calculus. We end the chapter with a complete axiom system for conformance, we also prove its soundness and completeness.

In chapter 4, we present some simple modelling examples in the fusion calculus. We present an abstract specification for a stack and then we give an implementation for stack. We also prove that the implementation is equivalent to the specification. As an exercise in modelling, we also present two models for a mobile phone network.

We conclude with our observations about the simplicity of the fusion calculus.

Chapter 2

Fusion Calculus

This chapter gives a brief introduction to the fusion calculus. We present its syntax and operational semantics using a labelled transition system. We also present the definitions for the bisimulation relations for the calculus. This chapter serves as a condensed version of the presentation given in [**?**]. All the definitions in this chapter are taken from [**?**] and we omit a reference to it, wherever convenient.

2.1 Syntax and Semantics

Like the π-calculus the fusion calculus postulates a set of names $\mathcal{N}$, ranged over by $u, v, \cdots, z$, used to represent communication ports, values, variables etc. In the fusion calculus the input primitive does not bind the names. The effect of an interaction is that the input and the output objects become identified in *fusions*. Formally, fusion is just an equivalence relation on names signifying that the names related by it should be considered equal. The effect of the fusion is regulated by the use of the scope. An interaction is written :

$$\overline{u}vw.P|uxy.Q \overset{\{v=x,w=y\}}{\longrightarrow} P|Q$$

Interactions in fusion calculus are symmetric, as can be seen by inverting the polarities.

$$uvw.P|\overline{u}xy.Q \overset{\{v=x,w=y\}}{\longrightarrow} P|Q$$

The only binding operator for the calculus is called scope. As is shown by the following example, scope can be used to delimit the extent of fusions:

$$\overline{u}vw.P|((x)(y)uxy.Q)|R|S \xrightarrow{\mathbf{1}} P|(Q\{v/x, w/y\}|R|S$$

Here **1** is the inert action, formally the identity fusion. We now give a formal definition for the syntax of the calculus.

2.1.1 Syntax

We assume an infinite set of names ranged over by $u, v, \cdots, z$. We write $\tilde{x}$ for a (possibly empty) finite sequence $x_1, x_2 \cdots x_n$. φ, ψ range over total equivalence relations on the set of names. If x and y are related by φ we write $x \varphi y$. We write $\{\tilde{x} = \tilde{y}\}$ to mean the smallest such equivalence relation relating each x_i with y_i.

Definition 2.1 *The free actions ranged over by α and the agents ranged over by $P, Q \cdots$ are defined by:*

$$
\begin{array}{rll}
\alpha ::= & u\tilde{x} & \text{(Input)} \\
 & \overline{u}\tilde{x} & \text{(Output)} \\
 & \varphi & \text{(Fusion)} \\
P ::= & \mathbf{0} & \text{(Inaction)} \\
 & \alpha.Q & \text{(Prefix)} \\
 & Q + R & \text{(Summation)} \\
 & Q|R & \text{(Composition)} \\
 & (x)Q & \text{(Scope)} \\
 & [x = y]Q & \text{(Match)} \\
 & [x \neq y]Q & \text{(Mismatch)} \\
 & (A(\tilde{x})) & \text{(Identifier)}
\end{array}
$$

The actions ranged over by γ, consist of the fusion actions and of communication actions of the form $(z_1) \cdots (z_n) a(\tilde{x})$ (written $(\tilde{z}) a\tilde{x}$) where $n \geq 0$ and all the elements in $\tilde{z}$ are also in $\tilde{x}$ Unlike the π-calculus the scope of $\tilde{x}$ is not bounded by the prefix action. We also define $\varphi \backslash z$ to mean $\varphi \cap (\mathcal{N} - \{z\})^2 \cup \{(z, z)\}$.

2.1.2 Semantics

The semantics for fusion calculus is again given by a structural congruence and a set of transition rules. The free names in P, written fn(P), are the names in P with a bound occurence. Note that in fusion calculus the prefix does not bind any any names. We write n(γ) to mean the set of all names in γ. Strucural congruence is the least congruence satisfying the rules given in table **??**.

The labelled transition system for fusion calculus is given by the least family of transition satisfying the laws in the table **??** The PREF action rule means the agent must perform a prefixed action before it can do anything else. Prefixing with a fusion represents an obligation to treat all the names related by the fusion as identical. The SUM action rule is for the choice

$P + Q \equiv Q + P$	$P + (Q + R) \equiv (P + Q) + R$
$P + \mathbf{0} \equiv P$	$P\vert Q \equiv Q\vert P$
$P\vert(Q\vert R) \equiv (P\vert Q)\vert R$	$P\vert\mathbf{0} \equiv P$
$(x)\mathbf{0} \equiv \mathbf{0}$	$(x)(y)P \equiv (y)(x)P$
$(x)(P + Q) \equiv (x)P + (x)Q$	$(x)MP \equiv M(x)P$ if $x \notin n(M)$
$P\vert(z)Q \equiv (z)(P\vert Q)$ where z $\notin fn(P)$	$A(\tilde{y}) \equiv P\{\tilde{y}/\tilde{x}\}$ if $A(\tilde{x}) \overset{\text{def}}{=} P$

Table 2.1: Structural conguence rules

operation. It indicates that the agent can choose one of the two summands. The PAR rule is for parallel execution of the two processes. COM results in a fusion action, meaning that the names made equal by the fusion are now the same. SCOPE effectuates the fusion by performing an appropriate substitution. It changes the action by removing the scoped name from the equivalence relation. PASS rule enables communication along a port which is not scoped. OPEN rule opens up the outermost scope. Leading to a bound action. This is the only rule relating to the bound actions. The rules MATCH and MISMATCH relate to the match and mismatch operators.

2.2 Bisimulation Relations

We present here the bisimulation relations for the calculus. Unlike the π-calculus we have only one bisimulation equivalence. First we give some definitions which will be used throughout the rest of the thesis.

Definition 2.2 *A substitution σ is idempotent if $\sigma\sigma = \sigma$*

Definition 2.3 *A substitution σ agrees with the fusion φ if $\forall x, y : x\varphi y \Leftrightarrow \sigma(x) = \sigma(y)$*

In other words, a substitution agrees with a fusion if it maps all names related by the fusion to the same name.

Definition 2.4 *A substitutive effect of a fusion φ is an idempotent substitution σ agreeing with φ. The only substitutive effect of a communication action is the identity substitution.*

A substitutive effect of a fusion is, therefore, a substitution that maps all members of each equivalence class in the fusion to one representative of the class.

PREF	$\dfrac{-}{\alpha.P \xrightarrow{\alpha} P}$
SUM	$\dfrac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'}$
PAR	$\dfrac{P \xrightarrow{\alpha} P'}{P\|Q \xrightarrow{\alpha} P'\|Q}$
COM	$\dfrac{P \xrightarrow{u\tilde{x}} P', Q \xrightarrow{\overline{u}\tilde{y}} Q',\ \ \|\tilde{x}\| = \|\tilde{y}\|}{P\|Q \xrightarrow{\{\tilde{x}=\tilde{y}\}} P'\|Q'}$
SCOPE	$\dfrac{P \xrightarrow{\varphi} P',\ \ z\varphi x,\ \ z \neq x}{(z)P \xrightarrow{\varphi \backslash z} P'\{x/z\}}$
PASS	$\dfrac{P \xrightarrow{\alpha} P',\ \ z \notin n(\alpha)}{(z)P \xrightarrow{\alpha} (z)P'}$
OPEN	$\dfrac{P \xrightarrow{(\tilde{y})a\tilde{x}} P',\ \ z \in \tilde{x} - \tilde{y},\ \ a \notin \{z, \overline{z}\}}{(z)P \xrightarrow{(z\tilde{y})a\tilde{x}} P'}$
MATCH	$\dfrac{P \xrightarrow{\alpha} P'}{[x = x] \xrightarrow{\alpha} P'}$
MISMATCH	$\dfrac{P \xrightarrow{\alpha} P'\ x \neq y}{[x \neq y] \xrightarrow{\alpha} P'}$

Table 2.2: Operational Semantics rules

2.2.1 Hyperequivalence

Definition 2.5 *A bisimulation is a binary symmetric relation S between agents such that $P\ S\ Q$ implies :*
If $P \xrightarrow{\gamma} P'$ with $bn(\gamma) \cap fn(Q) = \phi$
then $Q \xrightarrow{\gamma} Q'$ and $P'\sigma\ S\ Q'\sigma$ for some substitutive effect σ of γ.
P is bisimilar to Q, written $P \dot{\sim} Q$, if $P\ S\ Q$ for some bisimulation S.

A fusion action represents an obligation to treat the names related by the fusion as being identical. Therefore, we relate P' and Q' only after making the appropriate substitutions. The substitutive effect represents the "effect" of an action. Here it does not matter which substitution we choose. For example the relation R given below is a hyperequivalence.

$$A = (\{x = y, y = z\}.(x|\overline{y})$$

$$B = \{x = y, y = z\}.(x.\overline{y} + \overline{y}.x + 1))$$

$$R = \{A, B, (z|\overline{z}z, z.\overline{z} + \overline{z}.z + 1), (\overline{z}, \overline{z}), (z, z), (0, 0)\}$$

Hyperequivalence is not a congruence, the standard example would suffice to illustrate the point. $x|\overline{y}$ is bisimilar to $x.\overline{y} + \overline{y}.x$, but prefixing both with $\{x = y\}$ results in agents which are not bisimilar. The largest congruence included in bisimilarity is obtained by closing the definition of bisimulation under all substitutions.

Definition 2.6 *A hyperbisimulation is a substitution closed bisimulation, i.e., a bisimulation S with the property that $P\ S\ Q$ implies $P\sigma\ S\ Q\sigma$ for any substitution σ. Two agents P and Q are hyperequivalent, written $P \sim Q$, if they are related by a hyperbisimulation.*

2.2.2 Weak Hyperequivalence

In this section, we present the weak equivalence, where the agents are considered equivalent if they are "observed" to be equivalent. First we define the weak transitions.

Definition 2.7 *Composition of two transitions $\circ$ is defined by $P(\xrightarrow{\gamma} \circ \xrightarrow{\gamma'})Q$ iff there exists and agent P' such that $P \xrightarrow{\gamma} P'$ and $P'\sigma_\gamma \xrightarrow{\gamma'} Q$, where σ_γ is a substitutive effect of γ.*

Definition 2.8 *Conjunction of two fusion φ and ψ, written $\varphi \wedge \psi$, is defined to be the least equivalence relation containing φ and ψ.*

The weak transition represents the observable content of a sequence of actions. However, fusion actions can be observed only through their substitutive effects. As they have no subjects and the environment cannor synchronize with them. Therefore, while defining the weak transition we allow it to have a a sequence of fusions with the aggregated substitutive effect as intended.

Definition 2.9 *The weak transition $\xRightarrow{\gamma}$ is defined as: $P \xRightarrow{\gamma} Q$ if for some $n \geq 0$, $P \xrightarrow{\gamma_1} \circ \cdots \circ \xrightarrow{\gamma_n} Q$ and one of the following holds*

1. *γ is a communication and $\gamma = \gamma_i$ for some i and $\gamma_j = \mathbf{1}$ for all $j \neq i$.*
2. *γ and all γ_i are fusions and $\gamma = \gamma_1 \wedge \cdots \wedge \gamma_n$. The empty conjunction being $\mathbf{1}$ $P \xRightarrow{\mathbf{1}} P$ holds for all P.*

Definition 2.10 *A weak bisimulation is a binary symmetric relation S between agents such that $P\ S\ Q$ implies:*
If $P \xrightarrow{\gamma} P'$ with $bn(\gamma) \cap fn(Q) = \phi$
then $Q \overset{\gamma}{\Rightarrow} Q'$ and $P'\sigma S Q'\sigma$ for some substitutive effect σ of γ. P is weakly bisimilar to Q, written $P \dot{\approx} Q$, if $P\ S\ Q$ for some weak bisimulation S.

Weak hyperequivalence in then obtained by closing under substitution.

Definition 2.11 *A weak hyperbisimulation is a substitution closed weak bisimulation S with the property that $P\ S\ Q$ implies $P\sigma S Q\sigma$ for any substitution σ. Two agents P and Q are weak hyperequivalent, written $P \approx Q$, if they are related by a weak hyperbisimulation.*

Weak hyperequivalence is not a congruence as it is not preserved under the summation operator. The standard example being that $a.\mathbf{0} \approx \mathbf{1}.a.\mathbf{0}$ but $a.\mathbf{0} + b.\mathbf{0} \not\approx \mathbf{1}.a.\mathbf{0} + b.\mathbf{0}$. To obtain a congruence, we require that the first transition of an agent must be matched by at least one transition of the other agent.

Definition 2.12 *Define $P \overset{\gamma}{\Longrightarrow}_{+} Q$ to mean $P \overset{\gamma}{\Longrightarrow} Q$ if γ is not the silent action, and $P \xrightarrow{1}\overset{\gamma}{\Longrightarrow} Q$ otherwise.*

Definition 2.13 *Two agents P and Q are weakly hypercongruent , written $P \approx_{+} Q$, iff for any substitution σ*

$$P\sigma \xrightarrow{\gamma} P' \textit{ and } \mathrm{bn}(\gamma) \cap \mathrm{fn}(\gamma) = \phi \textit{ implies } Q\sigma \overset{\gamma}{\Longrightarrow}_{+} Q' \textit{ and } P'\sigma_\gamma \approx Q'\sigma_\gamma$$

where σ_γ, is a substitutive effect of γ, and vice-versa.

In the next chapter we present a similar treatment for the efficiency relations.

Chapter 3

Eficienc

Definition 3.3 *If R is an elaboration, then the closure of R is a binary relation* $S = \cup\{R\sigma : \sigma \text{ is an injective substitution}\}$

We now prove a property of the substitutions.

Lemma 3.1 *Any substitution* σ *can be defined as a composition of an idempotent and an injective substitution.*

Proof. Let σ be any substitution, we show by construction that it can be defined as a composition of an idempotent and an injective substitution.
If σ is idempotent or injective, the construction is trivial. σ can be defined as a composition of itself and the identity substitution.
If σ is non-idempotent and non-injective, there exists a sequence of names $x_1, x_2, \cdots, x_n$ such that $\sigma(x_1) = x_2, \sigma(x_2) = x_3, \cdots, \sigma(x_{n-1}) = x_n$. Where $x_1, x_2, \cdots, x_n$ are all distinct.
Construction.

1. Pick all maximal sequences and make them cyclic using a fresh name. If $x_1, x_2, \cdots, x_n$ is such a sequence and t is fresh then $\sigma'(x_n) = t$, $\sigma'(t) = x_1$, $\sigma'(x_1) = \sigma(x_1), \cdots, \sigma'(x_{n-1}) = \sigma(x_{n-1})$
2. For the remaining such sequences, drop all the variables that have been already included in σ'
3. Repeat steps 1 and 2 until there are no more sequences
4. For all x not in any such cycle $\sigma'(x) = x$. σ' is an injective substitution.
5. Let $\rho(t) = x_n$ for all fresh names introduced in σ'. ρ is idempotent. σ can be defined as $\sigma' \circ \rho$.

In what follows we prove various properties of the elaboration and its closure. We also define a largest elaboration relation and prove that it is a preorder.

Proposition 3.2 *Closure of an elaboration R is an elaboration*

Proof. Let R be a fusion elaboration and S be its closure. Let $(p, q) \in S$, if $(p, q) \in R$ we have nothing to prove, since $R \subset S$. So let $(p, q) \in R\sigma$ for some injective substitution σ.
$\Rightarrow\ \exists\, p_1, q_1$ such that $(p_1, q_1) \in R$ and $p_1\sigma = p,\ q_1\sigma = q$. Note, that if σ is injective then $p \xrightarrow{\gamma} p' \iff p\sigma \xrightarrow{\gamma\sigma} p'\sigma$.
$\Rightarrow$

$$1.\ p_1 \xrightarrow{\gamma} p_1' \Rightarrow \exists q_1' : q_1 \overset{\gamma}{\Longrightarrow} q_1' \wedge (p_1'\sigma_\gamma, q_1'\sigma_\gamma) \in R$$

$$2.\ q_1 \xrightarrow{\gamma} q_1' \Rightarrow \exists p_1' : p_1 \overset{\gamma}{\Longrightarrow}_+ p_1' \wedge (p_1'\sigma_\gamma, q_1'\sigma_\gamma) \in R$$

$\Rightarrow$

$$1.\ p_1\sigma \xrightarrow{\gamma\sigma} p_1'\sigma \Rightarrow \exists q_1' : q_1\sigma \overset{\gamma\sigma}{\Longrightarrow} q_1'\sigma \wedge (p_1'\sigma_\gamma\sigma, q_1'\sigma_\gamma\sigma) \in R\sigma$$

$$2.\ q_1\sigma \xrightarrow{\gamma\sigma} q_1'\sigma \Rightarrow \exists p_1' : p_1\sigma \overset{\gamma\sigma}{\Longrightarrow} p_1'\sigma \wedge (p_1'\sigma_\gamma\sigma, q_1'\sigma_\gamma\sigma) \in R\sigma$$

$\Rightarrow$

$$1.\ p \xrightarrow{\gamma} p' \Rightarrow \exists q' : q \overset{\gamma}{\Longrightarrow} q' \wedge (p'\sigma_\gamma, q'\sigma_\gamma) \in S$$

$$2.\ q \xrightarrow{\gamma} q' \Rightarrow \exists p' : p \overset{\gamma}{\Longrightarrow}_+ p' \wedge (p'\sigma_\gamma, q'\sigma_\gamma) \in S$$

We can make the last statement since, $\sigma_\gamma \circ \sigma \circ \sigma_\gamma = \sigma_\gamma$ and therefore, $p'\sigma_\gamma = p_1'\sigma_\gamma$. This finally proves the assertion.

Proposition 3.3 *If S_1 and S_2 are two closures, then $S_1 \circ S_2$ is also a closure.*

Proof. Let $S = S_1 \circ S_2$ and $(p,q) \in S$, then $\exists\ r : (p,r) \in S_1$ and $(r,q) \in S_2$

$$p \xrightarrow{\gamma} p' \Rightarrow \exists r' : r \overset{\gamma}{\Longrightarrow} r' \wedge p'\sigma S_1 r'\sigma \tag{3.1}$$

$$r \xrightarrow{\gamma} r' \Rightarrow \exists p' : p \overset{\gamma}{\Longrightarrow}_+ p' \wedge p'\sigma S_1 r'\sigma \tag{3.2}$$

where σ is some substitutive effect of γ, and

$$r \xrightarrow{\gamma} r' \Rightarrow \exists q' : q \overset{\gamma}{\Longrightarrow} q' \wedge r'\sigma S_2 q'\sigma \tag{3.3}$$

$$q \xrightarrow{\gamma} q' \Rightarrow \exists r' : r \overset{\gamma}{\Longrightarrow}_+ r' \wedge r'\sigma S_2 q'\sigma \tag{3.4}$$

We do a case analysis on the action γ to prove that S is a fusion elaboration.
Case 1. γ is a communication action, which implies that the only substitutive effect of γ is the identity substitution and that $\overset{\gamma}{\Longrightarrow}$ is the same as $\overset{\gamma}{\Longrightarrow}_+$. From equation 1 we have

$$p \xrightarrow{\gamma} p' \Rightarrow \exists r' : r \overset{\gamma}{\Longrightarrow} r' \wedge p'\sigma S_1 r'\sigma$$

now $r \overset{\gamma}{\Longrightarrow} r'$ is the same as $r \xrightarrow{\mathbf{1}} r_1^0 \xrightarrow{\mathbf{1}} r_1^1 \cdots \xrightarrow{\mathbf{1}} r_1^n \xrightarrow{\mathbf{1}} r_1 \xrightarrow{\gamma} r_2 \xrightarrow{\mathbf{1}} r_2^0 \xrightarrow{\mathbf{1}} r_2^1 \cdots \xrightarrow{\mathbf{1}} r_2^n \xrightarrow{\mathbf{1}} r'$. Now, if $r \xrightarrow{\mathbf{1}} r_1^0$

$$\exists q_1^0 : q \overset{\mathbf{1}}{\Longrightarrow} q_1^0 \ \wedge\ r_1^0 S_2 q_1^0$$

$$\exists q_1^1 : q_1^0 \overset{\mathbf{1}}{\Longrightarrow} q_1^1 \ \wedge\ r_1^1 S_2 q_1^1$$

$$\vdots$$

$$\exists q_1^n : q_1^{(n-1)} \overset{\mathbf{1}}{\Longrightarrow} q_1^n \ \wedge\ r_1^n S_2 q_1^n$$

and therefore, we can say that

$$\exists q_1 : q_1^n \overset{\mathbf{1}}{\Longrightarrow} q_1 \ \wedge\ r_1 S_2 q_1$$

and

$$\exists q_1 : q \overset{\mathbf{1}}{\Longrightarrow} q_1 \ \wedge \ r_1 S_2 q_1$$

now $r_1 \xrightarrow{\gamma} r_2$ implies

$$\exists q_2 : q_1 \overset{\mathbf{1}}{\Longrightarrow} q_2 \ \wedge \ r_2 S_2 q_2$$

quite similarly to the one proved above, we can prove that since $r_2 \overset{\mathbf{1}}{\Longrightarrow} r'$

$$\exists q' : q_2 \overset{\mathbf{1}}{\Longrightarrow} q' \ \wedge \ r' S_2 q'$$

$$\exists q' : q \overset{\mathbf{1}}{\Longrightarrow} q' \ \wedge \ r' S_2 q'$$

from this we conclude that

$$p \xrightarrow{\gamma} p' \Rightarrow \exists q' : q \overset{\gamma}{\Longrightarrow} q' \wedge p' S_1 r' \wedge r' S_2 q'$$

$$p \xrightarrow{\gamma} p' \Rightarrow \exists q' : q \overset{\gamma}{\Longrightarrow} q' \wedge p' S q'$$

and similarly for the other side.
Case 2 This is the trickiest part, which requires the use of the fact that S is closed for injective substitutions. γ is a fusion action, not the silent fusion. From equation 1 we have

$$p \xrightarrow{\gamma} p' \Rightarrow \exists r' : r \overset{\gamma}{\Longrightarrow} r' \wedge p' \sigma S_1 r' \sigma$$

$r \overset{\gamma}{\Longrightarrow} r'$ is the same as $r \xrightarrow{\varphi_1} \circ \xrightarrow{\phi_2} \circ \cdots \circ \xrightarrow{\varphi_n} r'$ for some $\phi_1, \phi_2, \cdots, \phi_n$ such that $\gamma = \varphi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n$. Let $r_1, r_2 \cdots r_{n-1}$ be the agents involved in the intermediate transitionsi and $\sigma_{\varphi_1}, \sigma_{\varphi_2}, \cdots, \sigma_{\phi_n}$ be their some substitutive effects. We prove the following claims first.
Claim 1. If σ_1 and σ_2 are the substitutive effects of the same fusion action then we can find an injective substitution ρ such that $\sigma_1 \circ \rho = \sigma_2$
Proof. We construct the substitution ρ as follows
If $\sigma_1(x) = \sigma_2(x) = t$ then $\rho(t) = t$. If $\sigma_1(x) = y$ and $\sigma_2(x) = z$, $y \neq z$, since σ_1 and σ_2 are the substitutive effects of the same fusion, say φ, $x\phi y$, $x\phi y$ and $y\varphi z$. $\Rightarrow$

$$\sigma_1(x) = \sigma_1(y) = \sigma_1(z) = y$$

and

$$\sigma_2(x) = \sigma_2(y) = \sigma_2(z) = z$$

Now define $\rho(y) = z$ and $\rho(z) = y$. The substitution ρ constructed in this manner in injective.
Claim 2. If $\varphi = \phi_1 \wedge \phi_2$ where ϕ, ϕ_1 and φ_2 are fusion actions, then for every substitutive effect σ of φ there exist σ_{ϕ_1} and σ_{ϕ_2} such that $\sigma = \sigma_{\varphi_1} \circ \sigma_{\phi_2} = \sigma_{\varphi_2} \circ \sigma_{\phi_1}$. σ_{ϕ_1} and σ_{ϕ_1} are substitutive effects of φ_1 and ϕ_2 respectively.

Proof. Proof follows from the fact that a fusion is an equivalence relation and its substitutive effect maps all the elements of an equivalence class to one member of the class. For each of the fusions φ_1 and ϕ_2, by choosing a suitable member as the representative of the class we can construct the substitutive effects, as required.
We prove by induction on the number of intermediate agents that, $\exists q' : q \overset{\gamma}{\Longrightarrow} q' \wedge r'\sigma_\gamma S_2 q'\sigma_\gamma$, where σ_γ is some substitutive effect of γ and we use the above claim to assert that $q'\sigma S_2 r'\sigma$
Basis. $n = 1$, $r \xrightarrow{\varphi_1} r' \Rightarrow \phi_1 = \gamma$ and therefore $\exists q' : q \overset{\gamma}{\Longrightarrow} q' \wedge r'\sigma_\gamma S_2 q'\sigma_\gamma$
Induction step From the induction hypothesis we have that

$$\exists q_{n-1} : q \overset{\gamma_{n-1}}{\Longrightarrow} q_{n-1} \wedge r_{n-1}\sigma_{\gamma_{n-1}} S_2 q_{n-1}\sigma_{\gamma_{n-1}}$$

where $\gamma_{n-1} = \varphi_1 \wedge \phi_2 \cdots \phi_{n-1}$ and $\sigma_{\gamma_{n-1}}$ is some substitutive effect of γ_{n-1}. From the definition of the weak transition, we have $r_{n-1}\sigma_{\varphi_{n-1}} \xrightarrow{\varphi_n} r'$ and from the sequence of transitions we can easily deduce that $r_{n-1}\sigma_{\varphi_{n-1}} S_2 q_{n-1}\sigma_{\phi_{n-1}}$ which implies $\exists q' : q_{n-1}\sigma_{\varphi_{n-1}} \overset{\phi_n}{\Longrightarrow} q' \wedge r'\sigma_{\varphi_n} S_2 q'\sigma_{\phi_n}$. Since all the substitutions are substitutive effects of some fusion actions, and the substitutive effects are idempotent we can say the following

$$r_{n-1}\sigma_{\gamma_{n-1}} \xrightarrow{\varphi_n\sigma_{\phi_1}\sigma_{\phi_2}\cdots\sigma_{\varphi_{n-2}}} r'\sigma_{\phi_1}\sigma_{\phi_2}\cdots\sigma_{\varphi_{n-2}}$$

$$r_{n-1}\sigma_{\gamma_{n-1}} \xrightarrow{\varphi_n\sigma_{\phi_1}\sigma_{\phi_2}\cdots\sigma_{\varphi_{n-1}}} r'\sigma_{\phi_1}\sigma_{\phi_2}\cdots\sigma_{\varphi_{n-1}}$$

$$r_{n-1}\sigma_{\gamma_{n-1}} \xrightarrow{\varphi_n\sigma_{\phi_1}\sigma_{\phi_2}\cdots\sigma_{\varphi_{n-1}}} r'\sigma_{\gamma_{n-1}}$$

and that

$$q_{n-1}\sigma_{\gamma_{n-1}} \overset{\varphi_n\sigma_{\phi_1}\sigma_{\phi_2}\cdots\sigma_{\varphi_{n-1}}}{\Longrightarrow} q'\sigma_{\phi_1}\sigma_{\phi_2}\cdots\sigma_{\varphi_{n-1}}$$

$$q_{n-1}\sigma_{\gamma_{n-1}} \overset{\varphi_n\sigma_{\phi_1}\sigma_{\phi_2}\cdots\sigma_{\varphi_{n-1}}}{\Longrightarrow} q'\sigma_{\gamma_{n-1}} \wedge r'\sigma_{\gamma_{n-1}}\sigma_{\varphi_n} S_2 q'\sigma_{\gamma_{n-1}}\sigma_{\phi_n}$$

which means $r'\sigma_{\gamma_n} S_2 q'\sigma_{\gamma_n}$ or $r'\sigma S_2 q'\sigma$
Case 3. γ is a the silent fusion action, which implies that the only substitutive effect of γ is the identity substitution. From equation 1 we have

$$p \xrightarrow{\gamma} p' \Rightarrow \exists r' : r \overset{\gamma}{\Longrightarrow} r' \wedge p'\sigma S_1 r'\sigma$$

Sub Case A. $r \overset{\mathbf{1}}{\Longrightarrow}_+ r'$ Proof is very similar to the one for Case 2. In this case γ is **1**
Sub Case B. $r \xrightarrow{\epsilon} r'$ i.e. $r \equiv r'$. Therefore, we have that $p'S_1 r$, we also have that rS_2q. Which implies $p'Sq$ and

$$p \xrightarrow{\gamma} p' \Rightarrow \exists q' \equiv q : q \overset{\mathbf{1}}{\Longrightarrow} q' \wedge p'Sq'$$

The first part is thus proved. The second part has the same proof as Sub Case A, because we only need to prove that $q \xrightarrow{\gamma} q' \Rightarrow \exists p' : p \overset{\mathbf{1}}{\Longrightarrow}_{+} p' \wedge p'Sq'$

Proposition 3.4 *The union of a family of fusion elaborations is a fusion elaboration.*

Proof. Consider $R = \cup R_i$ where R_i are the fusion elaborations. $(p, q) \in S \Rightarrow (p, q) \in S_i$ for some $i \Rightarrow$

$$\mathbf{1}.\ p \xrightarrow{\gamma} p' \Rightarrow \exists q' : q \overset{\gamma}{\Longrightarrow} \wedge p'\sigma S_i q'\sigma$$

$$2.\ q \xrightarrow{\gamma} q' \Rightarrow \exists p' : p \overset{\gamma}{\Longrightarrow}_{+} \wedge p'\sigma S_i q'\sigma$$

$\Rightarrow$

$$\mathbf{1}.\ p \xrightarrow{\gamma} p' \Rightarrow \exists q' : q \overset{\gamma}{\Longrightarrow} \wedge p'\sigma S q'\sigma$$
$$2.\ q \xrightarrow{\gamma} q' \Rightarrow \exists p' : p \overset{\gamma}{\Longrightarrow}_{+} \wedge p'\sigma S q'\sigma$$

Theorem 3.5 $\lesssim^{*}$ $= \cup\{$*R:R is an elaboration*$\}$ *is the largest elaboration.*

Proof. By proposition 4 $\lesssim^{*}$ is a fusion elaboration. If $\lesssim^{*}$ were not the largest fusion elaboration, let S be the largest fusion elaboration. Existence of S is guaranteed because there exists an upper bound on the binary relations on process agents. If S is a fusion elaboration $S \subseteq\ \lesssim^{*}$, so $\lesssim^{*}$ is at least as large as the largest fusion elaboration. Which is a contradiction. Therefore, $\lesssim^{*}$ is the largest fusion elaboration.

Theorem 3.6 $\lesssim^{*}$ *= closure of* $\lesssim^{*}$*, i.e. the family of elaborations is closed under injective substitution.*

Proof. This follows from Propositions 2 and 5

Theorem 3.7 $\lesssim^{*}$ *is a preorder*

Proof. Reflexivity follows from the fact that identity is a fusion elaboration. Transitivity follows from the propositions 6 and 3.

$\lesssim^{*}$ is not a precongruence for the same reasons due to which weak hyperequivalence is not a congruence. In fact, the same example suffices in this case also. Now, in the manner of weak hyperbisimulation we define a hyperelaboration.

Definition 3.4 *A hyperelaboration is a substitution closed elaboration, i.e. if S is a hyperelaboration then* $P\ S\ Q \Rightarrow P\sigma\ S\ Q\sigma$ *for any subsitution* σ*.* $\lesssim$ *is the union of all hyperelaborations.*

We now prove that the relation defined above is close under scope and composition, this is used in the next section to prove that conformance is a precongruence.

Theorem 3.8 $\lesssim$ *is closed under scope and composition.*

Proof. We construct an elaboration realtion to prove the proposition

1. *Scope* We first prove that $S = \{((x)P, (x)Q) : P \lesssim Q\} \cup \lesssim$ is a hyper-elaboration.

   Consider P and Q such that $P \lesssim Q$. If for some σ, $((x)P)\sigma \xrightarrow{\gamma} P'$, then we have the following possibilities.

   (a) γ is a free action and $\sigma(x) \notin fn(\gamma)$. In which case we have that $P\sigma \xrightarrow{\gamma} P''$, $P' \equiv (\sigma(x))P''$, and $\exists Q'' : Q\sigma \overset{\gamma}{\Longrightarrow} Q'' \wedge P''\sigma_\gamma \lesssim Q''\sigma_\gamma$. Therefore, we have that

   $$((x)Q)\sigma \overset{\gamma}{\Longrightarrow} (\sigma(x))Q'' \wedge (((\sigma(x))P'')\sigma_\gamma, ((\sigma(x))Q'')\sigma_\gamma) \in S$$

   (b) γ is a fusion action and for some fusion φ, $P\sigma \xrightarrow{\varphi} P''$, $\sigma(x)\phi z$, $z \neq \sigma(x)$ and $\gamma \equiv \varphi \backslash \sigma(x)$. In which case, $((x)P)\sigma \xrightarrow{\gamma} P''\{z/\sigma(x)\}$. We can now say that, $Q\sigma \overset{\varphi}{\Longrightarrow} Q''i \wedge P''\sigma_\phi \lesssim Q''\sigma_\phi$ which implies, $((x)Q)\sigma \overset{\gamma}{\Longrightarrow} Q''\{z/\sigma(x)\}$. Since, $\sigma_\varphi = \{z/\sigma(x)\} \circ \sigma_\gamma$ we can further say that $P''\{z/\sigma(x)\}\sigma_\gamma S Q''\{z/\sigma(x)\}\sigma_\gamma$.

   (c) γ is a communication action and $P\sigma \xrightarrow{(\tilde{y})a\tilde{z}} P'$, $\sigma(x) \in \tilde{z} - \tilde{y}$, $a \notin \{\sigma(x), \tilde{z}\}$, in which case we have $((x)P)\sigma \xrightarrow{(\sigma(x)\tilde{y})a\tilde{z}} P'$. Since, $Q\sigma \overset{(\tilde{y})a\tilde{z}}{\Longrightarrow} Q'$, we have $((x)Q)\sigma \overset{(\sigma(x)\tilde{y})a\tilde{z}}{\Longrightarrow} Q'$ and since σ_γ is the identity substitution we have $P'\sigma_\gamma S Q'\sigma_\gamma$

2. *Composition* We first prove that $S = \{(P|R, Q|R) : P \lesssim Q\} \cup \lesssim$ is a hyper elaboration.

   Consider P and Q such that $P \lesssim Q$. If for some σ, $(P|R)\sigma \xrightarrow{\gamma} P'$, then we have the following possibilities.

   (a) $P\sigma \xrightarrow{\gamma} P''$ and $P' \equiv P''|R\sigma$. and $\exists Q'' : Q\sigma \overset{\gamma}{\Longrightarrow} Q'' \wedge P''\sigma_\gamma \lesssim Q''\sigma_\gamma$. Therefore, we have that

   $$(Q|R)\sigma \overset{\gamma}{\Longrightarrow} Q''|R\sigma \wedge ((P''|R\sigma)\sigma_\gamma, (Q''|R\sigma)\sigma_\gamma) \in S$$

   (b) $R\sigma \xrightarrow{\gamma} R''$ and $P' \equiv P\sigma|R''$. Since $\lesssim$ is substitution closed, we have that

   $$(Q|R)\sigma \overset{\gamma}{\Longrightarrow} Q\sigma|R'' \wedge ((P\sigma|R'')\sigma_\gamma, (Q\sigma|R)\sigma_\gamma) \in S$$

(c) γ is a fusion and $P\sigma \xrightarrow{a\tilde{x}} P''$, $R\sigma \xrightarrow{\overline{a}\tilde{y}} R''$, $\gamma = \{\tilde{x} = \tilde{y}\}$. In which case we have $Q\sigma \xRightarrow{a\tilde{x}} Q'' \wedge P''\lesssim Q''$, $(Q|R)\sigma \xRightarrow{\gamma} Q''|R''$ and $Q' \equiv Q''|R''$. Since, $\lesssim$ is substitution closed we can say that $P'\sigma_\gamma S Q'\sigma_\gamma$

3.2 Conformance

In this section we define conformance and prove that it is a preorder preserved under all operators of the calculus. The definition is very similar to the one given for weak hypercongruence [**?**].

Definition 3.5 *Two agents P and Q are related by a fusion conformance, (written as $P \lesssim^+ Q$) iff for any substitution σ the following holds*

$$P\sigma \xrightarrow{\gamma} P' \text{ and } bn(\gamma) \cap fn(Q) = \varphi \Rightarrow Q\sigma \xRightarrow{\gamma}_+ Q' \text{ and } P'\sigma_\gamma \lesssim Q'\sigma_\gamma$$

where σ_γ is a substitutive effect of γ and vice-versa

Theorem 3.9 *$P\lesssim^+ Q$ iff for all R, $P + R\lesssim^+ Q + R$*

Proof. $\Rightarrow$ Given that $P\lesssim^+ Q$. Consider γ such that $bn(\gamma) \cap fn(Q + R) = \varphi$ and $(P + R)\sigma \xrightarrow{\gamma} P'$ that is $(P\sigma + R\sigma) \xrightarrow{\gamma} P'$. Then either $P\sigma \xrightarrow{\gamma} P'$ in which case, since $P\lesssim^+ Q$ we have.

$$Q\sigma \xRightarrow{\gamma}_+ Q' \wedge P'\sigma_\gamma \lesssim Q'\sigma_\gamma$$

$$Q\sigma + R\sigma \xRightarrow{\gamma}_+ Q' \wedge P'\sigma_\gamma \lesssim Q'\sigma_\gamma$$

$$(Q + R)\sigma \xRightarrow{\gamma}_+ Q' \wedge P'\sigma_\gamma \lesssim Q'\sigma_\gamma$$

or $R\sigma \xrightarrow{\gamma} P'$

$$R\sigma \xRightarrow{\gamma}_+ P'$$

$$Q\sigma + R\sigma \xRightarrow{\gamma}_+ P'$$

$$(Q + R)\sigma \xRightarrow{\gamma}_+ P' \wedge P'\sigma_\gamma \lesssim P'\sigma_\gamma$$

in either case we get the desired result. The vice-versa case can be proved similarly.

$\Leftarrow$ Given that $P + R\lesssim^+ Q + R$. Assume that $P \not\lesssim Q \quad \Rightarrow \exists\sigma, \gamma$ and P' such that $P\sigma \xrightarrow{\gamma} P'$ but if $Q\sigma \xRightarrow{\gamma}_+ Q'$ then $\neg(P'\sigma_\gamma \lesssim Q'\sigma_\gamma)$ Consider $R\sigma = \gamma.\mathbf{0}$ such that γ is a visible action not in $P\sigma$ or $Q\sigma$ Consider $R\sigma = \varphi.\mathbf{0}$ such that

ϕ is a visible action not in $P\sigma$ or $Q\sigma$. In the case when $\gamma \neq \mathbf{1}$ we have the following

$$P\sigma \xrightarrow{\gamma} P'$$

$$P\sigma + R\sigma \xrightarrow{\gamma} P'$$

since $P + R \lesssim^{+} Q + R$ we have

$$Q\sigma + R\sigma \overset{\gamma}{\Longrightarrow}_{+} Q'' \wedge P'\sigma_{\gamma} \lesssim Q''\sigma_{\gamma}$$

$$Q\sigma \overset{\gamma}{\Longrightarrow}_{+} Q'' \wedge P'\sigma_{\gamma} \lesssim Q''\sigma_{\gamma}$$

which is a contradiction. So, $P \lesssim^{+} Q$

Lemma 3.10 *Conformance is an elaboration and $P \lesssim^{+} Q$ implies $P \lesssim Q$*

Proof. Obvious, by definition. Since, $P \overset{\gamma}{\Longrightarrow}_{+} Q \Rightarrow P \overset{\gamma}{\Longrightarrow} Q$

Theorem 3.11 *Conformance is preserved under scope, choice, prefix, composition, match and mismatch operators*

Proof.

1. *Composition* Given that $P \lesssim^{+} Q$, consider

$$(P|R)\sigma \xrightarrow{\gamma} P'|R' \Rightarrow P\sigma|R\sigma \xrightarrow{\gamma} P'|R' \text{ and } bn(\gamma) \cap fn(Q) = \varphi$$

Case 1. $P\sigma \xrightarrow{\gamma} P'$ and $R\sigma \equiv R'$

$$\Rightarrow \quad Q\sigma \overset{\gamma}{\Longrightarrow}_{+} Q' \wedge P'\sigma_{\gamma} \lesssim Q'\sigma_{\gamma}$$

follows from theorem **??**

$$Q\sigma|R\sigma \overset{\gamma}{\Longrightarrow}_{+} Q'|R' \wedge P'\sigma_{\gamma}|R'\sigma'_{\gamma} \lesssim Q'\sigma_{\gamma}|R'\sigma_{\gamma}$$

$$(Q|R)\sigma \overset{\gamma}{\Longrightarrow}_{+} Q'|R' \wedge (P'|R')\sigma'_{\gamma} \lesssim (Q'|R')\sigma_{\gamma}$$

Case 2. $R\sigma \xrightarrow{\gamma} R'$ and $P\sigma \equiv P'$

$$\Rightarrow \quad R\sigma \overset{\gamma}{\Longrightarrow}_{+} R' \wedge R'\sigma_{\gamma} \lesssim R'\sigma_{\gamma}$$

let, $Q' \equiv Q\sigma$. It follows from theorem **??** and lemma **??** and the definition of the relation $\lesssim$

$$Q\sigma|R\sigma \overset{\gamma}{\Longrightarrow}_{+} Q'|R' \wedge (P\sigma)\sigma_{\gamma}|R'\sigma_{\gamma} \lesssim (Q\sigma)\sigma_{\gamma}|R'\sigma_{\gamma}$$

$$(Q|R)\sigma \overset{\gamma}{\Longrightarrow}_{+} Q\sigma|R' \wedge (P'|R')\sigma'_{\gamma} \lesssim (Q'|R')\sigma_{\gamma}$$

Case 3. $P\sigma \xrightarrow{u(\tilde{x})} P'$, $R\sigma \xrightarrow{\overline{u}(\tilde{y})} R'$ and $\gamma \equiv \{x = y\}$

$$\Rightarrow \quad Q\sigma \overset{\gamma}{\Longrightarrow}_{+} Q' \wedge P' \lesssim Q'$$

$$Q\sigma \xrightarrow{\mathbf{1}} Q_0 \xrightarrow{\mathbf{1}} \cdots \xrightarrow{\mathbf{1}} Q_{k-1} \xrightarrow{u\tilde{x}} Q_k \xrightarrow{\mathbf{1}} \cdots \xrightarrow{\mathbf{1}} Q_n \xrightarrow{\mathbf{1}} Q'$$

$Q\sigma|R\sigma \xrightarrow{\mathbf{1}} Q_0|R\sigma \xrightarrow{\mathbf{1}} \cdots \xrightarrow{\mathbf{1}} Q_{k-1}|R\sigma \xrightarrow{u\tilde{x}} Q_k|R' \xrightarrow{\mathbf{1}} \cdots \xrightarrow{\mathbf{1}} Q_n|R' \xrightarrow{\mathbf{1}} Q'|R'$

Also, $P' \lesssim Q' \Rightarrow P'|R' \lesssim Q'|R' \Rightarrow (P'|R')\sigma_\gamma \lesssim (Q'|R')\sigma_\gamma$. Thus in either case we have the required.

2. *Choice* The proof is very similar to the cases 1 and 2 for composition

3. *Scope* Given $P \lesssim^{+} Q$. If for some σ, $((x)P)\sigma \xrightarrow{\gamma} P'$, then we have the following possibilities.

   (a) γ is a free action and $\sigma(x) \notin fn(\gamma)$. In which case we have that $P\sigma \xrightarrow{\gamma} P''$, $P' \equiv (\sigma(x))P''$, and $\exists Q'' : Q\sigma \overset{\gamma}{\Longrightarrow}_{+} Q'' \wedge P''\sigma_\gamma \lesssim Q''\sigma_\gamma$. Therefore, from theorem **??** we have that

   $$((x)Q)\sigma \overset{\gamma}{\Longrightarrow}_{+} (\sigma(x))Q'' \wedge ((\sigma(x))P'')\sigma_\gamma \lesssim ((\sigma(x))Q'')\sigma_\gamma$$

   (b) γ is a fusion action and for some fusion φ, $P\sigma \xrightarrow{\varphi} P''$, $\sigma(x)\phi z$, $z \neq \sigma(x)$ and $\gamma \equiv \varphi \backslash \sigma(x)$. In which case, $((x)P)\sigma \xrightarrow{\gamma} P''\{z/\sigma(x)\}$. We can now say that, $Q\sigma \overset{\varphi}{\Longrightarrow}_{+} Q'' \wedge P''\sigma_\phi \lesssim Q''\sigma_\phi$ which implies, $((x)Q)\sigma \overset{\gamma}{\Longrightarrow}_{+} Q''\{z/\sigma(x)\}$. Since, $\sigma_\varphi = \{z/\sigma(x)\} \circ \sigma_\gamma$ we can further say that $P''\{z/\sigma(x)\}\sigma_\gamma \lesssim Q''\{z/\sigma(x)\}\sigma_\gamma$.

   (c) γ is a communication action and $P\sigma \xrightarrow{(\tilde{y})a\tilde{z}} P'$, $\sigma(x) \in \tilde{z} - \tilde{y}$, $a \notin \{\sigma(x), \tilde{z}\}$, in which case we have $((x)P)\sigma \xrightarrow{(\sigma(x)\tilde{y})a\tilde{z}} P'$. Since, $Q\sigma \overset{(\tilde{y})a\tilde{z}}{\Longrightarrow}_{+} Q'$, we have $((x)Q)\sigma \overset{(\sigma(x)\tilde{y})a\tilde{z}}{\Longrightarrow}_{+} Q'$ and since σ_γ is the identity substitution we have that $P'\sigma_\gamma \lesssim Q'\sigma_\gamma$

4. *Prefix* Given that $P \lesssim^{+} Q$, consider $\alpha.P$

   $$(\alpha.P)\sigma \xrightarrow{\alpha\sigma} P\sigma$$

   $$(\alpha.Q)\sigma \overset{\alpha\sigma}{\Longrightarrow}_{+} Q\sigma$$

   and $P \lesssim^{+} Q \Rightarrow P \lesssim Q \Rightarrow P\sigma\sigma_{\alpha\sigma} \lesssim Q\sigma\sigma_{\alpha\sigma}$. Which is the required condition.

5. *Match and mismatch* Given that $P \lesssim^+ Q$, consider $[x = y]P$ and $[x = y]Q$, now if $\sigma(x) = \sigma(y)$ then both the agents have a γ derivative which satisfies the required conditions. If $\sigma(x) \neq \sigma(y)$ then neither of them has any action. So, in either case we have the desired. A similar reasoning holds for the case of mismatch.

Theorem 3.12 *Fusion conformance is preserved under recursive definition*

Proof. Let E and F be two process expressions where the process expressions are defined in the usual way and the various functions and operations are extended to the set of process expressions in the usual way. Let us assume that E $\lesssim^+$ F. We now prove that if p and q are two processes defined as $p \overset{def}{=} E\{p/x\}$ and $q \overset{def}{=} F\{q/x\}$. We prove that $p \lesssim^+ q$. Consider the relation

$$S = \{(g\{p/x\}, g\{q/x\})\text{: where g has atmost x as the free process variable}\}$$

We claim that for all substitutions σ the relation S satisfies the following.

$$g\{p/x\}\sigma \xrightarrow{\gamma} p' \Rightarrow \exists q' : g\{q/x\}\sigma \xrightarrow{\gamma} q' \wedge p'\sigma_\gamma S \circ \lesssim q'\sigma_\gamma \tag{3.5}$$

$$g\{q/x\}\sigma \xrightarrow{\gamma} q' \Rightarrow \exists p' : g\{p/x\}\sigma \xrightarrow{\gamma} q' \wedge p'\sigma_\gamma \lesssim \circ\, S q'\sigma_\gamma \tag{3.6}$$

Where σ_γ is some substitutive effect of γ We prove the claim .by induction on the depth of the inferences $g\{p/x\}\sigma \xrightarrow{\gamma} p'$ and $g\{q/x\}\sigma \xrightarrow{\gamma} q'$

1. $g \equiv x$ for some process variable x. Then, $g\{p/x\} = p$. If $p \xrightarrow{\gamma} p'$ then by a shorter derivation we can say that $e\{p/x\} \xrightarrow{\gamma} p'$ and so

$$\exists q' : f\{q/x\} \xrightarrow{\gamma} q' \wedge p'\sigma_\gamma S \circ \lesssim q'\sigma_\gamma$$

$$\exists q' : q \xrightarrow{\gamma} q' \wedge p'\sigma_\gamma S \circ \lesssim q'\sigma_\gamma$$

$$\exists q' : g\{q/x\} \xrightarrow{\gamma} q' \wedge p'\sigma_\gamma S \circ \lesssim q'\sigma_\gamma$$

2. $g \equiv \alpha.Q$ and $\gamma \equiv \alpha\sigma$ then $g\{p/x\}\sigma = \alpha\sigma.Q\{p/x\}\sigma$ and $p' \equiv Q\{p/x\}\sigma$. Also, $g\{q/x\}\sigma \equiv \alpha\sigma.Q\{q/x\}\sigma \xrightarrow{\alpha\sigma} Q\{q/x\}\sigma$. It is easy to see now that $Q\{p/x\}\sigma\sigma_\gamma S \circ \lesssim Q\{q/x\}\sigma\sigma_\gamma$

3. $g \equiv Q + R$ and $g\{p/x\}\sigma \equiv Q\{p/x\}\sigma + R\{p/x\}\sigma$. Now, if $g\{p/x\}\sigma \xrightarrow{\gamma} p'$ then either $Q\{p/x\}\sigma \xrightarrow{\gamma} p'$ or $R\{p/x\}\sigma \xrightarrow{\gamma} p'$. If former is the case then we have

$$Q\{q/x\}\sigma \xrightarrow{\gamma} q' \wedge p'\sigma\sigma_\gamma S \circ \lesssim q'\sigma\sigma_\gamma$$

otherwise

$$R\{q/x\}\sigma \xrightarrow{\gamma} q' \wedge p'\sigma\sigma_\gamma S \circ \lesssim q'\sigma\sigma_\gamma$$

in either case we have that

$$g\{q/x\}\sigma \xrightarrow{\gamma} q' \wedge p'\sigma\sigma_\gamma S \circ \lesssim q'\sigma\sigma_\gamma$$

4. $g \equiv Q|R$ There are three possibilities, in this case. The first two cases are very similar to the case above. We just need to use theorem **??**. Consider the third case when, $\gamma = \{\tilde{y} = \tilde{z}\}$, $Q\{p/x\}\sigma \xrightarrow{u(\tilde{y})} Q_1$ and $R\{p/x\}\sigma \xrightarrow{\overline{u}(\tilde{z})} R_1$. Then

$$Q\{q/x\}\sigma \xrightarrow{u(\tilde{y})} Q_2 \wedge Q_1\sigma_\gamma S \circ \lesssim Q_2\sigma_\gamma$$

$$R\{q/x\}\sigma \xrightarrow{\overline{u}(\tilde{x})} R_2 \wedge R_1\sigma_\gamma S \circ \lesssim R_2\sigma_\gamma$$

and thus

$$g\{q/x\}\sigma \xrightarrow{\gamma} Q_2|R_2$$

Now,

$$Q_1\sigma_\gamma S \circ \lesssim Q_2\sigma_\gamma \Rightarrow \exists Q' : Q_1\sigma_\gamma S Q' \lesssim Q_2\sigma_\gamma$$
$$R_1\sigma_\gamma S \circ \lesssim R_2\sigma_\gamma \Rightarrow \exists R' : R_1\sigma_\gamma S R' \lesssim R_2\sigma_\gamma$$

by definition of S we must have that for some Q_1' and R_1'

$$Q_1\sigma_\gamma = Q_1'\{p/x\}$$
$$Q' = Q_1'\{q/x\}$$
$$R_1\sigma_\gamma = R_1'\{p/x\}$$

and

$$R' = R_1'\{q/x\}$$

Therefore,

$$Q_1\sigma_\gamma|R_1\sigma_\gamma = (Q_1'|R_1')\{p/x\}$$
$$Q'|R' = (Q_1'|R_1')\{q/x\}$$

and

$$(Q_1|R_1)\sigma_\gamma S(Q'|R')$$

Also it follows from theorem **??** and $\varphi\alpha\psi axiom\alpha\alpha\alpha\alpha\alpha\psi\alpha\alpha s$ $transitiv\psi a\psi Fr\alpha Cp\psi Fu\alpha\alpha\alpha p\psi Ft\alpha$