CSL105: Discrete Mathematical Structures
I semester 2008-09
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Tutorial sheet: **Well-orderings, Countability and Uncountability**

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1. Let X and Y be any (finite, countably infinite or uncountably infinite) sets such that X is not equipollent to Y (i.e. $X \not\approx Y$) but there exists a total injective function from Y to X. Prove that there *does not exist any total injective function* from X to Y.

2. Let $(A, \leq_A)$ and $(B, \leq_B)$ be two well-founded sets. Prove that their cartesian product is also well-founded. What is the ordering on $A \times B$?

3. Consider the lexicographic ordering on $\mathbb{N}^2$ defined by $(m, n) \prec (p, q)$ iff $m < p$ or ($m = p$ and $n < q$) and the partial order $\preceq$ defined by $(m, n) \preceq (p, q)$ iff $(m, n) \prec (p, q)$ or $(m, n) = (p, q)$.

   (a) Prove that $\preceq$ well-orders $\mathbb{N}^2$.

   (b) How will you define a lexicogrphic ordering which well-orders $\mathbb{N}^i$ for any given $i \geq 0$?

   (c) Prove that $\mathbb{N}^* = \bigcup_{i \geq 0} \mathbb{N}^i$ is also well-ordered by an ordering that preserves the lexicographic ordering on $\mathbb{N}^i$ for each i.

4. Prove the following **principle of well-ordered induction**.

   Let $(A, \leq)$ be well-ordered set. Let $B \subseteq A$ be a set such that

   Basis The least element of A is in B,

   Induction-step For any $a \in A$, if for all $b \in A$, $b < a$ implies $b \in B$, then $a \in B$.

   Then $B = A$.

5. Let $(A, \leq)$ be a well-ordered set and let $B \subseteq A$ be a set such that $f : A \to B$ is an order-preserving isomorphism

   (a) Prove that for all $a \in A$, $a \leq f(a)$.

   (b) If $S_b = \{c \in A | c < b\}$ for any $b \in A$, prove that there is no isomorphism between A and S_b.

6. Let $(A, \leq)$ be a well-ordered set. For each $a \in A$ define $S_a = \{b \in A | b < a\}$.

   (a) Prove that A is not isomorphic to any S_a.

   (b) Prove that $(A, \leq)$ is order isomorphic to $(\mathscr{S}_A, \subseteq)$, where $\mathscr{S}_A = \{S_a | a \in A\}$

   (c) If for each $a \in A$, S_a is isomorphic to an ordinal, then A is isomorphic to an ordinal.

7. Prove that the following sets are countably infinite.

   (a) The set $\mathbb{N}^k$ of k-tuples of natural numbers for any $k > 1$.

   (b) The set $\mathbb{Z}$ of integers.

   (c) The set $\mathbb{Q}$ of rational numbers.

   (d) The set A^* of finite sequences of elements from a *finite nonempty* set A.

   (e) The set A^* of finite sequences of elements from a *countably infinite* set A.

   (f) The set Exp of arithmetic expressions involving only natural numbers and the operators for addition and multiplication.

8. Prove that

   (a) $|\{f \mid f : \mathbb{N} \to \mathbb{N}\}| = \aleph_1$

   (b) $|\{f \mid f : \mathbb{R} \to \mathbb{R}\}| = \aleph_2$

9. Prove the following:

   (a) Any infinite subset of a countably infinite set is also countably infinite.

   (b) Let $\{A_i \mid i \in \mathbb{N}\}$ be a collection of mutually disjoint *countably infinite* sets. Then $\bigcup_{i\in\mathbb{N}} A_i$ is a countably infinite set.

   (c) Use the above result to show that the set of all tuples of naturals is countably infinite.

10. From the following theorem it trivially follows that the set of irrationals is uncountably infinite. Let A be an uncountably infinite set and let $B \subset A$ be countably infinite. Then $C = A \backslash B$ is an uncountably infinite set.