CSL105: Discrete Mathematical Structures
I semester 2008-09
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Tutorial sheet: **Partial orders, Lattices and Boolean Algebras**

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1. (a) Prove that for any irreflexive-transitive binary relation $<$ on a nonempty set A, the relation $\leq$ on A defined by $a \leq b$ iff $a = b$ or $a < b$ is a partial order on A.

   (b) Conversely prove that if $\leq$ is a partial order on a nonempty set A, then $\leq -Id_A$ is an irreflexive-transitive relation on A.

2. Let $(A, \leq_A)$ and $(B, \leq_B)$ be partial orders. Prove that $(A \times B, \leq_{AB})$ and $(A \times B, \leq_{lex})$ are also partial orders, where

   - $\leq_{AB}$ is the *pointwise ordering* defined by $(a_1, b_1) \leq_{AB} (a_2, b_2)$ iff $a_1 \leq_A a_2$ and $b_1 \leq_B b_2$ and
   - $\leq_{lex}$ is the *lexicographic ordering* defined by $(a_1, b_1) \leq_{lex} (a_2, b_2)$ iff $a_1 <_A a_2$ or $(a_1 = a_2$ and $b_1 \leq_B b_2)$

3. A partially ordered set $(A, \leq)$ is said to be *dense* if for all $a, b \in A$, $a < b$ implies there exists $c \in A$ such that $a < c < b$. Prove that a dense poset with at least two distinct comparable elements is not well-founded.

4. Prove that the set of nonempty strings of lower-case roman characters under the lexicographic ordering is neither dense nor well-founded.

5. A *linear extension* of a partial order $\leq$ on a set is a total order $\leq_t$ such that $\leq \subseteq \leq_t$. Prove that

   (a) every finite partial order has at least one linear extension and

   (b) every finite partial order is the intersection of all its linear extensions.

   (c) Give an algorithm to construct a linear extension of a finite partial order.

6. A company has a finite collection P of projects and a finite totally ordered set R of security restrictions ordered by $<$. Consider the set of security classes defined by $R \times \mathbf{2}^P$. Information is permitted to flow from one security class $S_1 = (r_1, Q_1)$ to another $S_2 = (r_2, Q_2)$ iff $r_1 \leq r_2$ and $Q_1 \subseteq Q_2$. Prove that the set of security classes is a lattice under the information flow ordering $(\sqsubseteq)$.

7. Let $\Pi(A)$ denote the set of all possible partitions of a non-empty set A, and let $\sqsubseteq$ denote the partition-refinement relation.

   (a) Show that $\sqsubseteq$ is a partial order.

   (b) Define the least upper bound (lub) and the greatest lower bound (glb) of a subset of Π. Prove that they are indeed the lub and the glb.

   (c) Prove that $(\Pi(A), \sqsubseteq)$ is a complete lattice.

8. Prove that the composition of two continuous functions on a complete lattice $(L, \sqsubseteq)$ yields another continuous function.

9. Let $\mathbb{Z}$ be the set of integers and let $\mathbb{Z}_\perp^\top = \mathbb{Z} \cup \{\perp, \top\}$ be the set extended with the two new elements $\perp$ and $\top$. Let $\sqsubseteq$ be the binary relation on $\mathbb{Z}_\perp^\top$ defined by $a \sqsubseteq b$ <u>iff</u> $a = b$ <u>or</u> $a \sqsubset b$, where $a \sqsubset b$ <u>iff</u> $(a = \perp$ <u>and</u> $b \neq \perp)$ <u>or</u> $(a \neq \top$ <u>and</u> $b = \top)$.

   (a) Prove that $(\mathbb{Z}_\perp^\top, \sqsubseteq)$ is a complete lattice by clearly defining the operations $\bigsqcup$ and $\bigsqcap$ for every nonempty subset of $(\mathbb{Z}_\perp^\top$.

   (b) Any partial function $f : \mathbb{Z} \mapsto \mathbb{Z}$ may be extended to the total function $f_\perp^\top : \mathbb{Z}_\perp^\top \to \mathbb{Z}_\perp^\top$ so that

   $$f_\perp^\top(a) = \begin{cases} \perp & \text{if } a = \perp \text{ or } f(a) \text{ is undefined} \\ \top & \text{if } a = \top \\ f(a) & \text{otherwise} \end{cases}$$

   Prove that $f_\perp^\top$ is monotonic in $(\mathbb{Z}_\perp^\top, \sqsubseteq)$.

(c) Give an example of a function on $\mathbb{Z}$ whose extension (as in the previous part) is <u>not</u> continuous.

10. Prove that in a complemented distributive lattice $(L, \sqsubseteq)$,

    (a) $\bot$ and $\top$ are complements of each other

    (b) the complementation operation is *antimonotonic* i.e. $a \sqsubseteq b$ implies $c(b) \sqsubseteq c(a)$.

11. A *homomorphism* between lattices $(L_1, \sqsubseteq_1)$ and $(L_2, \sqsubseteq_2)$ is a total function $h : L_1 \rightarrow L_2$ such that the following properties hold:

    - $h(\top_1) = \top_2$
    - $h(\bot_1) = \bot_2$
    - $h(a_1 \sqcup_1 b_1) = h(a_1) \sqcup_2 h(b_1)$
    - $h(a_1 \sqcap_1 b_1) = h(a_1) \sqcap_2 h(b_1)$

    (a) Prove that a homomorphism is a monotonic function.

    (b) Prove that not all monotonic functions are homomorphisms.