COL351: Analysis and Design of Algorithms

Ragesh Jaiswal, CSE, IITD

Applications of Network Flow

- Given a weighted directed graph representing a transportation network.
- There are multiple supply nodes in the graph denoting the places that has a factory for some product.
- There are multiple demand nodes denoting the consumption points.
- Each supply node v has an associated supply value $s(v)$ denoting the amount the product it can supply.
- Each demand node v has a similar demand value $d(v)$.
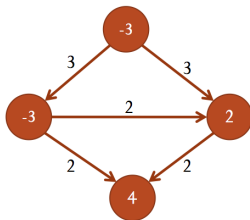- Question: Is there a way to ship product such that all demand and supply goals are met?

Problem

Given a directed graph G with integer edge capacities. For each node v, there is an associated demand value $t(v)$ denoting the demand at the node (*for supply nodes this is* $-s(v)$, *for demand nodes* $d(v)$, *for other nodes* 0). Find whether there exists a flow f such that for all nodes v:

$$f^{in}(v) - f^{out}(v) = t(v)$$

and the capacity constraints are met. Such a flow is called a feasible circulation.

Problem

Given a directed graph G with integer edge capacities. For each node v, there is an associated demand value $t(v)$ denoting the demand at the node (*for supply nodes this is $-s(v)$, for demand nodes $d(v)$, for other nodes* 0). Find whether there exists a flow f such that for all nodes v:

$$f^{in}(v) - f^{out}(v) = t(v)$$

and the capacity constraints are met. Such a flow is called a feasible circulation.

- <u>Claim 1</u>: For a feasible circulation to exist, $\sum_v t(v) = 0$. (*That means supply equals the demand*)

- <u>Claim 1</u>: For a feasible circulation to exist, $\sum_v t(v) = 0$. (*That means supply equals the demand*)
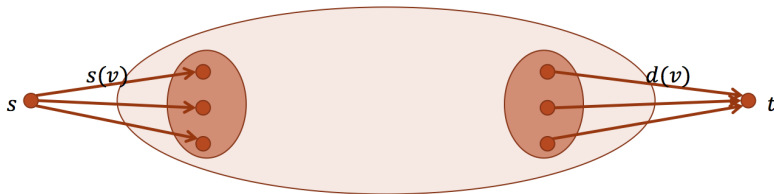- Consider the flow network as shown in the diagram below and let $D = \sum_{\text{demand node } v} d(v)$.

Figure: Connect source to supply nodes and demand nodes to sink.

- <u>Claim 1</u>: For a feasible circulation to exist, $\sum_v t(v) = 0$. (*That means supply equals the demand*)
- Consider the flow network G' as shown in the diagram below and let $D = \sum_{\text{demand node } v} d(v)$.
- <u>Claim 2</u>: There is a feasible circulation in G iff the maximum flow in the network G' is D.
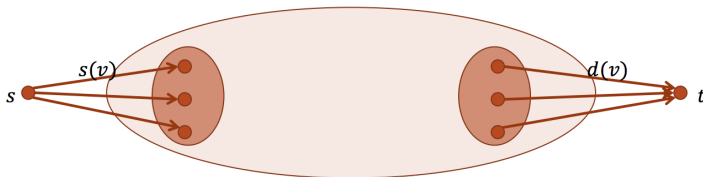

Figure: Connect source to supply nodes and demand nodes to sink.

- <u>Claim 1</u>: For a feasible circulation to exist, $\sum_v t(v) = 0$. (*That means supply equals the demand*)
- Consider the flow network G' as shown in the diagram below and let $D = \sum_{\text{demand node } v} d(v)$.
- <u>Claim 2</u>: There is a feasible circulation in G iff the maximum flow in the network G' is D.
  - (if) Consider the max-flow and remove s, t.
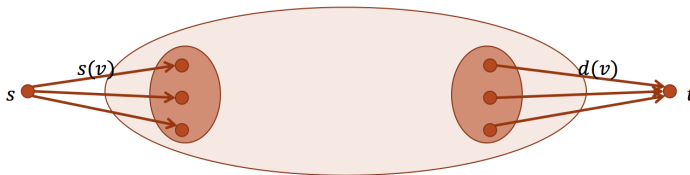 - (only if) Extend the feasible circulation in the network.

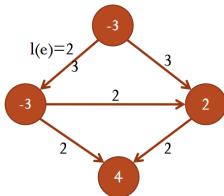Figure: Connect source to supply nodes and demand nodes to sink.

Problem

Given a directed graph G with integer edge capacities $c(e)$ and lower bounds $l(e)$. For each node v, there is an associated demand value $t(v)$ denoting the demand at the node (*for supply nodes this is $-s(v)$, for demand nodes $d(v)$, for other nodes* 0). Find whether there exists a flow f such that for all nodes v:

$$f^{in}(v) - f^{out}(v) = t(v)$$

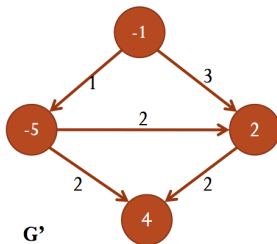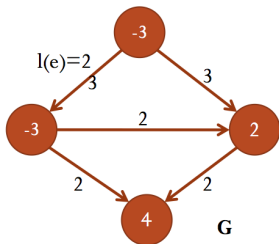and the following capacity constraints are met. For every edge e:
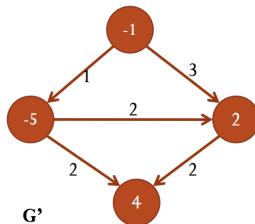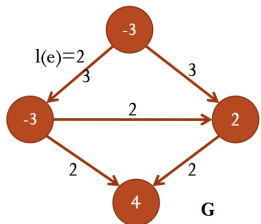
$$l(e) \leq f(e) \leq c(e)$$

- Consider a flow f such that for all edge e, $f(e) = l(e)$.
- For each vertex v, let $r(v) = f^{in}(v) - f^{out}(v)$.
- Construct a new graph G':
  - Each edge e in G' has capacity $c(e) - l(e)$.
  - Each vertex v in G' has a demand $t(v) - r(v)$.
- <u>Idea</u>: Solve the feasible circulation problem without lower bounds on G'.

Network Flow
## Feasible Circulation with Lower Bounds

- Consider a flow f such that for all edge e, $f(e) = l(e)$.
- For each vertex v, let $r(v) = f^{in}(v) - f^{out}(v)$.
- Construct a new graph G':
    - Each edge e in G' has capacity $c(e) - l(e)$.
    - Each vertex v in G' has a demand $t(v) - r(v)$.
- <u>Idea</u>: Solve the feasible circulation problem without lower bounds on G'.
- <u>Claim</u>: There is a feasible circulation (with lower bounds) in G iff there is a feasible circulation in G'.
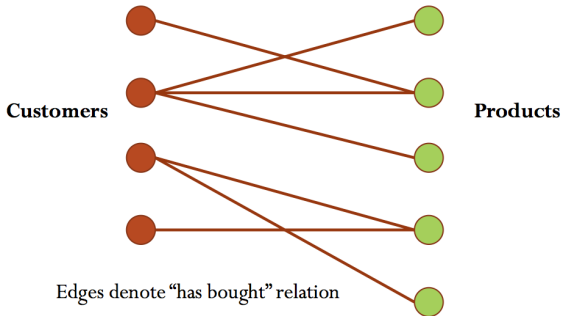

G

G'

- <u>Claim</u>: There is a feasible circulation (with lower bounds) in G iff there is a feasible circulation in G'.
  - (**if**) Let f' be a feasible circulation in G'. Consider f where $f(e) = f'(e) + l(e)$. Is f a feasible circulation in G?
  - (**only if**) Let f be a feasible circulation in G. Consider f' where $f'(e) = f(e) - l(e)$. Is f' a feasible circulation in G'?
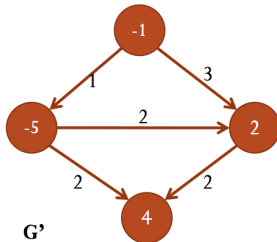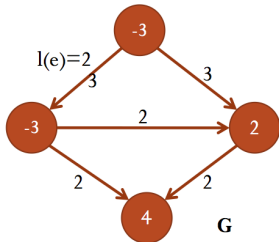
Network Flow
Survey Design

Problem

There are n customers and m products. Each customer i is supposed to review between $c(i)$ and $c'(i)$ products that he has bought in the past and each product j should be reviewed by between $p(j)$ and $p'(j)$ customers. Find a way to do the survey.

Customers

Products

Edges denote "has bought" relation

Problem

There are n customers and m products. Each customer i is supposed to review between $c(i)$ and $c'(i)$ products that he has bought in the past and each product j should be reviewed by between $p(j)$ and $p'(j)$ customers. Find a way to do the survey.
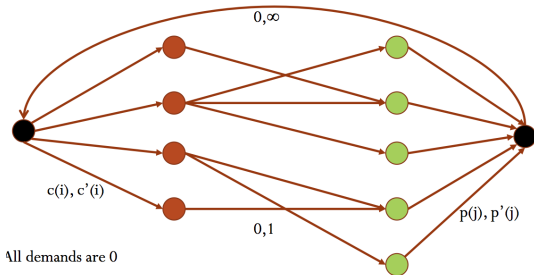
- Consider the flow network set up below.
- <u>Claim</u>: The survey is feasible iff there is a feasible circulation (with lower bounds) in the network.

0,∞

c(i), c'(i)

p(j), p'(j)

0,1

All demands are 0

Definition (Edge-disjoint path)

Two paths P_1 and P_2 between from vertex s to vertex t are called edge-disjoint if P_1 and P_2 do not share any edges.

Problem

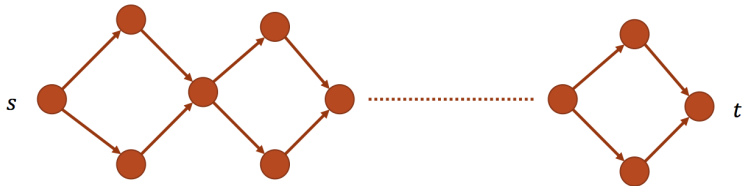Given an unweighted directed graph G, find the maximum number of edge-disjoint paths between s and t in G.

> **Problem**
>
> Given an unweighted directed graph G, find the maximum number of edge-disjoint paths between s and t in G.

- How many paths from s to t are present in this graph?
- How many edge-disjoint paths from s to t are present in this graph?
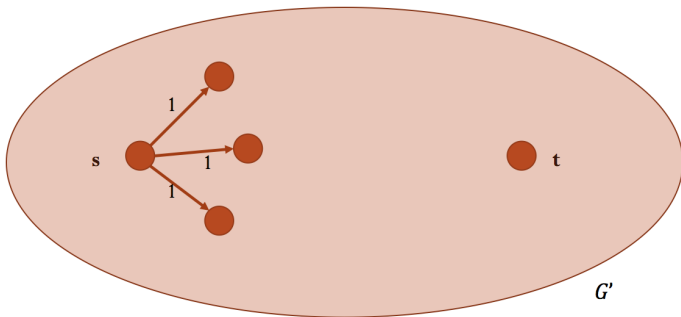
Problem

Given an unweighted directed graph G, find the maximum number of edge-disjoint paths between s and t in G.
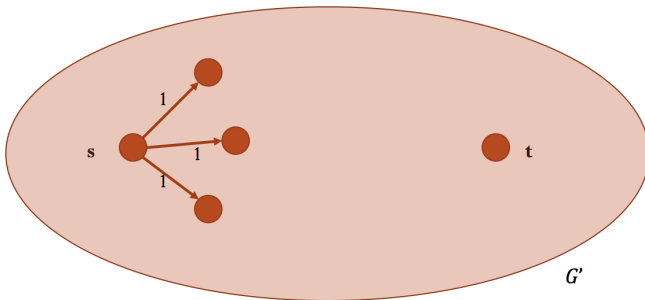
- Consider the network G' below:

Problem

Given an unweighted directed graph G, find the maximum number of edge-disjoint paths between s and t in G.

- <u>Claim 1</u>: If there are k edge-disjoint paths in G, then there is an s-t flow in the network with value at least k.

G'

> **Problem**
>
> Given an unweighted directed graph G, find the maximum number of edge-disjoint paths between s and t in G.

- <u>Claim 1</u>: If there are k edge-disjoint paths in G, then there is an s-t flow in the network with value at least k.
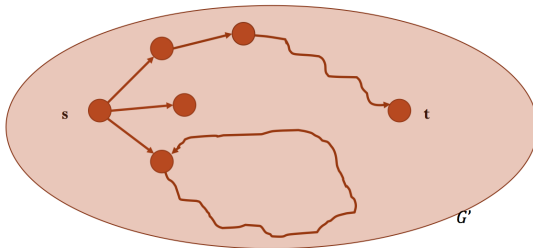- <u>Claim 2</u>: If there is an s-t flow in G' of value k, then there are at least k edge-disjoint paths in G.
  - <u>*Idea*</u>: Use induction on the number of edges with non-zero flow value.

End