COL872:Expander Graphs	May 1, 2015

Partitioning Into Expanders:Gharan and Trevisan

Dhiraj Madan *Entry No.: 2011CS50545*

1 Review

Earlier we have seen some of the spectral properties of a graph.To review for a d-regular graph G=(V,E) with adjacency matrix A,we defined a normalised adjacency matrix M=$\frac{A}{d}$. We also defined a notion of expansion of a cut $(S, \bar{S})$ as h(S)=$\frac{E(S,\bar{S})}{min|S|,|\bar{S}|}$.We denoted the eigen values of M by $\lambda_1, \lambda_2, ..., \lambda_n$. It was then shown that:-

1. $1 = \lambda_1 \geq \lambda_2 \geq ...\lambda_n \geq -1$
2. $\lambda_2 = 1$ iff G is disconnected
3. $\lambda_n = -1$ iff G is bipartite

We also proved a softer version of statement 2 as cheeger's inequality i.e.
Cheeger's Inequality:
$\frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)}$
Note: The same results also hold for a weighted graph which is not d regular.In that case we define define a matrix of degrees D,with D_{ii}=degree of vertex i=$\sum_j w(i,j)$ and $D_{ij} = 0 \forall i \neq j$.Then M is given by L=$D^{-1/2}AD^{-1/2}$.It may be noted that $M = \frac{A}{d}$ for a d-regular graph.The above results however hold in general for the generalised definition of M.In this case we define the expansion as $\frac{w(S,\bar{S})}{vol(S)}$, where $vol(S) = \sum_{u \in S} \sum_{vinV} w(u,v).h(G) = min_{S:vol(S) \leq \frac{vol(S)}{2}} h(S)$ In today's discussion we will generalise this setting in several ways:-

1. We will consider a generalised definition of a partition into several components which may be larger than 2.We will have to generalise the notion of expansion factor for the same.
2. Secondly we will also generalise the definition to ensure that the number of edges to break 2 different components of parttition is small,but the same to break a single component of partition is large

To remain consistent with paper we define a notion of a laplacian matrix.

Definition 1 ***Laplacian****For an undirected weighted graph G=(V,E), with weights a weight function $w : VXV \to \mathbb{R}^+$ (w(u,v)=0 if (u,v) is not an edge), define a matrix of degrees D,with D_{ii}=degree of vertex i=$\sum_j w(i,j)$ and $D_{ij} = 0 \forall i \neq j$.Then laplacian of G is given by L=$I - D^{-1/2}AD^{-1/2}$..e. L=I-M.*

The eigen values of L and M have a simple relation that is:-

Lemma 1 *v is an eigen vector of M associated with eigen value λ iff v is an eigen vector of L associated with eigen value* $1-\lambda$

Proof. $\implies$
$Mv = \lambda v \implies (I-M)v = v - \lambda v \implies Lv = (1-\lambda)v$
$\impliedby$
$Lv = (1-\lambda)v \implies (I-M)v = v - \lambda v \implies Mv = \lambda v$

■

Denote λ'_i as the i^{th} **smallest** eigen value of L. The previous result can now be stated in terms of L as:-

1. $0 = \lambda'_1 \leq \lambda'_2 \leq ...\lambda'_n \leq 2$
2. $\lambda'_2 = 0$ iff G is disconnected
3. $\lambda'_n = 2$ iff G is bipartite

Cheeger's Inequality:$\frac{\lambda'_2}{2} \leq h(G) \leq \sqrt{2(\lambda'_2)}$

2 Generalising the result to large number of partitions

The previous result can be generalised to the case of more than 2 disconnected components as well.

Theorem 2 *If L is the laplacian of a graph G, then multiplicity of 0 as an eigen value of L=Number of maximally discnnected components of G.*

Proof. Let the number of disconnected components of G be k.Order the vertices so that vertices in same maximally connected component are numbered consecutively.
Let A be the adjacency matrix in this numbering.(Note that the above operation is allowed, since permuting the rows and columns of a matrix ensures that the matrices remain similar.In this case,the eigen values of the matrix do not change.)
By choice of ordering the matrix can now be partitioned into k blocks along the diagonal each corresponding to a maximally connected component of G.Say A=

$$\begin{pmatrix} A_1 & 0 & 0.. \\ 0 & A_2 & 0.. \\ 0 & 0 & .. \\ 0 & 0 & ..A_k \end{pmatrix}$$

Now f(x)=characteristic polynomial of A
=det(xI-A)=$det(xI - A_1)det(xI - A_2)...det(xI - A_k)$.
(By determinant of block decomposition) But $det(xI - A_i)$=characteristic polynomial of i^{th} maximally connected subgraph=$f_i(x)$(let).
Now $f(x) = \Pi_{i=1}^{k} f_i(x)$.
Now since each A_i is connected, the degree of x in $f_i(x)$=algebraic multiplicity of eigen value 0 in $f_i(x)$=1
Thus algebraic multiplicity of 0 in f(x) is k($x^k | f(x)$ and x^{k+1} does not divide f(x)).
Thus geometric multiplicity of 0 as an eigen value of f(x)=algebraic multiplicity of 0 as an eigen value of f(x)=k. ∎

Corollary 1 *For a graph G=(V,E) with eigen values of laplacian(in non decreasing order) as* λ'_i,$\lambda'_k = 0$ *iff G has at least k disconnected components.*

Proof. λ'_k=0 iff multiplicity of 0 as an eigen value $\geq$k.(Follows from the fact that λ'_i form a non decreasing sequence and $\lambda'_1 = 0$). But multiplicity of 0 as an eigen value =Number of maximally connected components.(Proved above).Thus λ'_k=0 iff G has at least k disconnected components. ∎

To state a generalised version of Cheeger's inequality, we need to generalise the definition of expansion factor.

Definition 2 $\rho(k)$*Define* $\rho(k) = min_{\ A_1,...,A_k,\ A_i \neq \phi,\ A_i \cap A_j \neq \phi, i \neq j} max h(A_i)$.

Note that A_i are only required to be disjoint and need not form a partition of V.Moreover though A_i can not be empty sets, the definition does not explicitly state $A_i \neq V$.This will implied when $k \geq 2$, but the ρ is well defined to k=1 as well.
Observations

1. $\rho(1) = 0$.Choose A_1=V.Now h(V)=0.

2. $\rho(2) = h(G)$.Though it appears intuitive ,the subtlety is that the definition of ρ does not restrict A_1, A_2 to be a partitioning,only the condition that they are disjoint suffices.
   Proof.

   - $\rho(2) \leq h(G)$
     Let $S \subset V$ that realises h(G) with $vol(S) \leq \frac{vol(V)}{2}$.Clearly $h(\bar{S}) \leq h(S)$(Since volume is larger but weights of edges going across is same).Thus $max(h(S), h(\bar{S})) = h(S)$.Since $\rho(2) = min_{A_1,A_2,disjoint\ and\ non\ empty}\ max(h(A_1), h(A_2))$
     $\leq max(max(h(S), h(\bar{S}))) = h(G)$.

- $h(G) \leq \rho(2)$
  Let A_1 and A_2 be the disjoint sets that realise $\rho(2)$.Now $vol(A_1) + vol(A_2) \leq vol(V)$.Thus one of these must have volume $\leq \frac{vol(V)}{2}$.
  WLOG assume $vol(A_1) \leq \frac{vol(V)}{2}$.
  Now $\rho(2) = max(h(A_1), h(A_2)) \geq h(A_1) \geq h(G)$.

  Thus $\rho(2) = h(G)$ ■

3. $0 = \rho(1) \leq \rho(2) \leq \rho(3)... \leq \rho(n)$

It turns that cheeger's inequality can be generalised in terms of ρ for any arbitrary k.

Theorem 3 (LOT12) *For any graph G,with λ'_i being the i^{th} smallest eigen value of its laplacian, and for an arbitrary $k \geq 2$, we have the following:-*
$\frac{\lambda'_k}{2} \leq \rho(k) \leq O(k^2)\sqrt{\lambda'_k}$.

3 Partitioning into expanders:Main Result

Ideally we would like to partition a graph into subsets of vertices such that it is easy to disconnect any 2 distinct subsets, but not so easy to disconnect any 1 subset.In order to mathematically define such a partitioning , we define a notion of (h_{in}, h_{out}) clustering.

Definition 3 *We say k disjoint subsets $A_1, ...A_k$ of V are a (h_{in}, h_{out}) clustering if for all $1 \leq i \leq k$, $h(G[A_i]) \geq h_{in}$ and $h(A_i) \leq h_{out}$*

Here $h(G[A_i])$ refers to the expansion factor of the subgraph induced on A_i(considering all valid cuts in this subgraph). On the other hand $h(A_i)$ considers a single cut A_i as a part of the original graph. The definition captures the requirement that it should be difficult to disconnect a valid subset, but easy 2 disconnect 2 distinct components.

Theorem 4 *If for a graph G, $\rho(k+1) > (1+\epsilon)\rho(k)$,for some $0 < \epsilon < 1$,then*

1. *There exists k disjoint subsets of V that form a $(\epsilon\rho(k+1)/7, \rho(k))$ clustering.*
2. *There exists a k partitioning of V that is a $(\epsilon\rho(k+1)/(14k), k\rho(k))$*

Note that the parts 1 and 2 of above theorem differ in the fact that a partitioning requires subsets whose union is V apart from being disjoint.

Corollary 2 *For any $k \geq 2$, if $\lambda_k > 0$ then for some $1 \leq l \leq k-1$, there is an l partitioning of V into sets $P_1, ...P_l$ that is a* $=(\Omega(\frac{\rho(k)}{k^2}), O(l\rho(l))) = ((\Omega(\frac{\lambda'_k}{k^2}), O(l^3\sqrt{\lambda_l})))$ *clustering.*

Proof. To prove the statement, we need to choose a value of l and ϵ carefully enough and apply the above theorem.
Since $\rho(k) \geq \frac{\lambda'_k}{2} > 0$ and $\rho(1) = 0, \exists l < k$ such that $\rho(l)(1+\frac{1}{k})$(Choosing l equal to largest index i such that $\rho(i)$=0 suffices).
Now choose largest such l<k satisfying $\rho(l)(1+\frac{1}{k})$.
Telescoping we obtain that $\rho(k) \leq (1+\frac{1}{k})^{k-l-1}\rho(l+1) \leq e.\rho(l+1)$.Now applying the second part theorem on l,$\exists$ a partitioning into sets $P_1, P_2, ...P_l$ such that $\forall 1 \leq i \leq l$, $h(G[P_i]) \geq \frac{\rho(l+1)}{14kl} \geq \frac{\rho(k)}{14k^2e} \geq \frac{\lambda_k}{28k^2e} = \frac{\lambda_k}{80k^2}$ and $h(P_i) \leq l\rho(l) \leq O(l^3)\sqrt{\lambda_l}$ ■

3.1 Proof idea for main Theorem

The first part of main theorem guarantees existence of a k disjoint sets each having $h(P_i) \leq O(\rho(k))$ and having an internal expansion $\geq \Omega\rho(k+1)$.
A "wrong" way to prove the result would be to argue that there exists k disjoint sets, say $P_1, ...P_l$ with $h(A_i) \leq \rho(k)$.
Now if we split any of these further we have a k+1 disjoint subsets and hence $h_{in} \geq \rho(k+1)$.
The fallacy is that on partitioning one of the sets say P_i into say B_1 and B_1 some of the edges from B_1 go outside P_i as well.
The proof could have "worked" if we could ensure that for any P_i,any proper subset of P_i ,say B, out of all edges which go outside B, a large fraction of them go into edges within P_i.
To state this formally we need a definition.

Definition 4 *For any set $P \subseteq V$ and $S \subset P$,define* $\psi(S,P) = \frac{w(S,P/S)}{w(S,V/P).\frac{vol(P-S)}{vol(P)}}$

The proof of part 1 works in 2 steps as follows:-

1. First it is shown that for a graph satisfying conditions of theorem and disjoint sets $A_1, ...A_k$,satisfying $h(A_i) \leq \rho(k)$ we can construct k sets $B_1, ..., B_k$ where $B_i \subseteq A_i$, $h(B_i) \leq h(A_i)$ and $\psi(S, B_i) \geq \frac{\epsilon}{3}$ for every $S \subset B_i$.
   This can be proved by starting from sets $B_i = A_i$(initial assignment) and arbitrarily picking setsB_i,S which violate the condition, and setting B_i=S or B_i/S whichever has lesser expansion.
   It can be proved that at each stage $h(B_i)$ decreases.
   Moreover $\psi(S, B_i) \geq \frac{\epsilon}{3}$ for every $S \subset B_i$ is ensured as a terminating condition.

2. Further it can be shown that if B_i satisfy the condition stated in above step, then $h(G[B_i]) \geq \frac{\epsilon\rho(k+1)}{7}$

The proof of second part works by first finding disjoint sets satisfying the first part of the theorem.Using the same we can extend it to a valid partitioning.
Initialise each $P_i = B_i$ for $i \leq k-1$ and $P_k = V - B_1 - B_2 - ...B_k$(where B_i's are disjoint subsets from first part).

Now we ensure that a large fraction of edges from $P_i - B_i$ enter P_i.
Formally we need $w(S, P_i/S) \leq \frac{w(S,V/S)}{k}$ for every $S \subseteq P_i - B_i$.

4 Conclusion

The paper also gives an algorithmic result for the same which is slightly weaker.
The paper has thus shown an existential result for a good (h_{in}, h_{out}) clustering for a constant gap between $\rho(k)$ and $\rho(k+1)$. It is suggested as an open problem if the analysis can be extended to the case when there is a constant gap between λ'_k and λ'_{k+1}.Currently this requires a larger gap.

References

[1] Gharan, Shayan Oveis, and Luca Trevisan. "Partitioning into expanders." Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms. SIAM, 2014. APA

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