COL870: Clustering Algorithms

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- How do we solve k-median (in metric space) approximately?
  - First Idea: Try writing a Linear Program (LP) for the discrete version of the problem.
  - Second Idea: Try a local search heuristic.
    1. Start with k centers $T \subset S$ chosen arbitrarily.
    2. At every step, replace a centre in T with a point in S given that the cost decreases due to this "swap".
    - We will argue that when the above local search algorithm terminates, we obtain a constant factor approximation.
    - For ease of discussion, we will discuss this heuristic for the k-means problem in Euclidean space and skip the running time analysis.

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Local search for k-means

- Initialize centers $T \subset S$ arbitrarily
- While $\exists t \in T$, $t' \in S$, such that $\Phi(T + \{t'\} - \{t\}) < \Phi(T)$
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Lemma

Let O be the subset of k data points that minimise $\Phi(O)$ for the discrete k-means problem.Let T be the solution returned by the above local search procedure. Then $\Phi(T) \leq 25 \cdot \Phi(O)$.

- This gives an approximation factor of 50 for the k-means problem.
- Project Topic: There is a much better analysis of this local search than what we see here. This procedure is also efficient.

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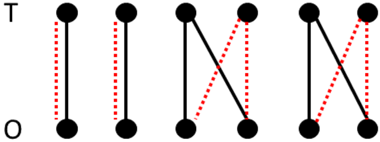
- Consider a bipartite graph with nodes corresponding to T and O on either side. There is an edge from a node $o \in O$ to a node $t \in T$ iff t is the nearest point in T to o.



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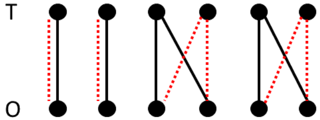

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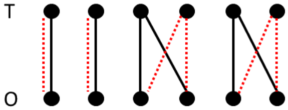

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- Putting together claims 4 and 5 gives us the result.

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    - A pseudo-approximation algorithm for the k-median problem outputs more than k centers and the approximation factor is computed w.r.t. the optimal solution with k centers.
  - Second Idea: Try a local search heuristic for the discrete version of the problem.

End