- **Use of unfair means will be severely penalized.**

There are 6 questions for a total of 20 points.

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(2) 1. For any language $L \subseteq \{0,1\}^*$ and any string $x \in \{0,1\}^*$, let $L(x) = 1$ if $x \in L$ and 0 otherwise. Consider two languages $L_1, L_2, L_3 \in \mathsf{NP}$ and consider the following two languages:

1. $L_{\oplus} = \{(x_1, x_2, x_3) : L_1(x_1) \oplus L_2(x_2) \oplus L_3(x_3) = 1\}$,
2. $L_{MAJ} = \{(x_1, x_2, x_3) : MAJ(L_1(x_1), L_2(x_2), L_3(x_3)) = 1\}$,

where $MAJ()$ denotes the majority function. Can you say that $L_{\oplus}, L_{MAJ} \in \mathsf{NP}$? Explain your answer.

(4) 2. Argue that the following language is in P^{NP}

$$\mathsf{USAT} = \{\psi | \ \psi \text{ is a boolean formula in 3CNF form that has one satisfying assignment}\}.$$

(3) 3. Define $\mathsf{P}_{/\log} = \cup_{c \geq 1} \mathsf{P}_{/c \log n}$, the class of languages decidable by Turing Machines running in polynomial time that take $O(\log n)$ bits of advice. Show that if $\mathsf{NP} \subseteq \mathsf{P}_{/\log}$, then $\mathsf{P} = \mathsf{NP}$.

4. A language $A \subseteq \{0,1\}^*$ is said to be *sparse* if for every n the number of strings of size at most n in A is at most n^c for some constant c. In the next few steps, we will show that a sparse set A is NP-hard if and only if $\mathsf{P} = \mathsf{NP}$.

(1) (a) Argue that $\mathsf{P} = \mathsf{NP}$ implies that sparse set A is NP-hard.

(1) (b) Consider the following language

$$L = \{\langle \psi, x \rangle \ : \ x \in \{0,1\}^n, \psi \text{ is a Boolean formula in } n \text{ variables s.t. } \psi \text{ is satisfiable and } x \succeq a \text{ as per lexicographical ordering, where } a \text{ denotes the smallest satisfying assignment}\}$$

Show that $L \in \mathsf{NP}$.

(4) (c) Consider any sparse set A. If A is NP-hard, then $L \leq_p A$. Let f denote the many-one mapping from L to A. Figure out a way to use f to solve SAT.
(Hint: For a satisfiable formula ψ, use $f(\psi, x_1), ..., f(\psi, x_m)$ for carefully chosen $x_1, ..., x_m$ to eliminate non-satisfying assignments of ψ).

(3) 5. In the lecture, we designed a circuit of depth $O(\log n)$ for computing the sun of two n-bit numbers. What was the size of this circuit? Design a circuit that has logarithmic depth and linear size.

(2) 6. Show that if $\mathsf{NP} \subseteq \mathsf{BPP}$, then $\mathsf{NP} = \mathsf{RP}$.