CSL 356: Analysis and Design of Algorithms
Instructor: Ragesh Jaiswal

Linear Programming (LP) A linear programming (henceforth LP) problem consists of assigning real values to variables such that these variables satisfy the following two properties:

1. **Linear Constraints**: The variables satisfy a set of linear equalities or inequalities.
2. **Objective function**: The variables maximise (or minimize) a given linear objective function.

Here is an example of a linear programming problem:

$$\begin{aligned} \text{Maximize } x_1 + 5 \cdot x_2, \\ \text{subject to } x_1 \geq 0 \\ x_2 \geq 0 \\ x_1 \leq 100 \\ x_2 \leq 200 \\ x_1 + x_2 \leq 250 \end{aligned}$$

Integer Linear Program (ILP): An Integer Linear Programming (henceforth ILP) problem is defined similarly as LP except that here we have the additional constraint that the variables are integers. Most of the NP problems that we discussed in the class can very easily be rephrased as an ILP problem.

- Model the Maximum Independent Set and Vertex Cover problems as ILP.
- Argue that the following problem is NP-complete:
  Decision-ILP: Given an integer vector C, an $n \times n$ integer matrix A, an integer vector b, and an integer k, determine if there exists an integer assignment $y_1, ..., y_n$ to variables $x_1, ..., x_n$ such that

$$\begin{aligned} C^T \cdot [y_1, ..., y_n] \geq k, \\ \text{subject to } A \cdot [y_1, ..., y_n] \leq b \end{aligned}$$

Approximation using LP: Consider the optimization version of the Vertex Cover problem. That is, given a graph $G = (V, E)$ find a vertex cover of G of minimum size. We know that this problem is NP-hard. Consider the following Integer LP for solving this problem: there are n variables, one for each vertex. Let the variable corresponding to v be denoted by x_v.

LP1

Minimize :	$\sum_{v \in V} x_v$,
subject to :	$x_u + x_v \geq 1$ for every edge $(u, v) \in E$
	$x_v \in \{0, 1\}$ for every vertex $v \in V$

We know that the optimal solution to the above Integer LP would give the minimum sized vertex cover of G. The problem in using this LP to solve the Minimum Vertex Cover problem is that Integer Linear Program is also NP-Hard. Suppose we relax the integer constraint in the above linear program. That is, we consider the following *relaxed* LP:

LP2

Minimize :	$\sum_{v \in V} x_v,$
subject to :	$x_u + x_v \geq 1$ for each edge $(u, v) \in E$
	$x_v \geq 0$ for every vertex $v \in V$

Show the following:

1. Let $(o_1, ..., o_n)$ be an optimal solution of LP2 (these are assignments to the variables). Argue that for all $v \in V, 0 \leq o_v \leq 1$.

2. Suppose we consider the following subset of vertices:

$$S = \{v : o_v \geq 1/2\}$$

Argue that S is a vertex cover of the graph G.

3. Argue that S is at most twice the size of the minimum vertex cover of G. Note that this shows that we have an approximation algorithm for the vertex cover problem that gives approximation factor of 2.

4. Show that the approximation factor of 2 is tight. Recall, we defined tightness when we talked about approximation algorithms using greedy technique.