CSL 356: Analysis and Design of Algorithms

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Topics

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming
- Network Flow
- Computational intractability
- <u>Other topics</u>: Randomized algorithms, Computational Geometry, Number-theoretic algorithms etc.

Computational Intractability

Introduction

Computational Intractability

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- There are a lot of problems arising in various fields for which this question is still unresolved.
- Question: Are these problems *related* in some manner? Are there certain aspects that are common to all these problems?
- Question: If someone discovers an efficient algorithm to one of these difficult problems, then does that mean that there are efficient algorithms for other problems? If so, how do we obtain such an algorithm.

Computational Intractability

- <u>NP-Complete Problems</u>: This is a large class of problems such that all problems in this class are equivalent in the following sense:
  - *A polynomial time algorithm for any one problem in this class implies the existence of polynomial time algorithm for all of them.*

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- Polynomial time reduction:
  - Consider two problems X and Y.
  - Suppose there is a *black box* that solves arbitrary instances of problem X.
  - Suppose any arbitrary instance of problem Y can be solved using a polynomial number of standard computational steps and a polynomial number of calls to the black box that solves instance of problem X correctly.
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- Is $Bipartite - matching \leq_p Max - flow$?

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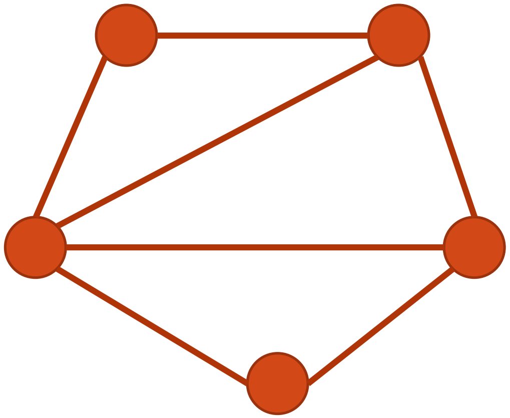
  - We say that Y is polynomial time reducible to X or $Y \leq_p X$.
- <u>Claim 1</u>: Suppose $Y \leq_p X$. If X can be solved in polynomial time, then Y can be solved in polynomial time.
- <u>Claim 2</u>: Suppose $Y \leq_p X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

Computational Intractability

Polynomial-time reductions

Computational Intractability: Reduction

- Problem(Independent set): Given a graph $G = (V, E)$ and an integer k, check if there is an *independent set* of size at least k in G.
  - *Independent set*: A subset I of vertices is called an independent set if there is no edge between any pair of vertices in I.
- Problem(Maximum independent set): Given a graph $G = (V, E)$, output the size of the *independent set* of G of maximum size.



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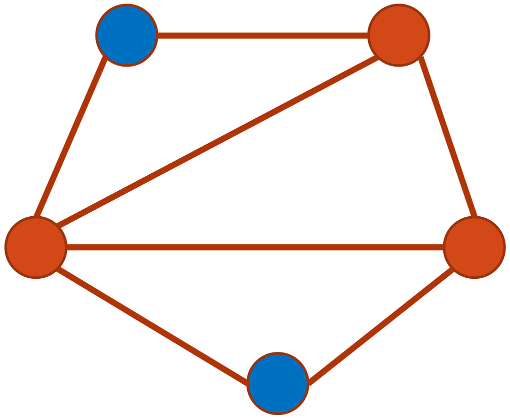

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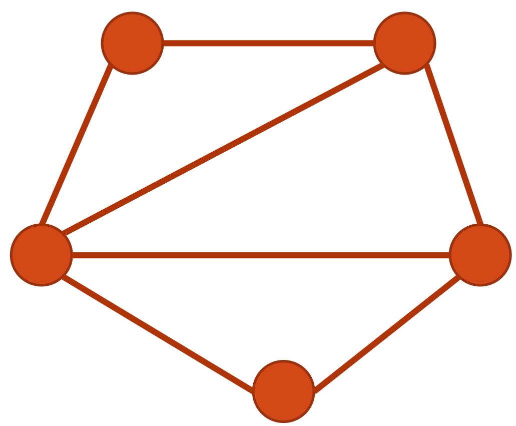
- <u>Problem(Maximum independent set)</u>: Given a graph $G = (V, E)$, output the size of the *independent set* of G of maximum size.

- <u>Claim 1</u>: Maximum-independent-set $\leq_p$ Independent-set
- <u>Claim 2</u>: Independent-set $\leq_p$ Maximum-independent-set

Computational Intractability: Reduction

- Problem(Vertex cover): Given a graph $G = (V, E)$ and an integer k, check if there is a *vertex cover* of size at most k in G.
  - *Vertex cover*: A subset S of vertices is called a vertex cover of G if for any edge (u, v) in the graph u is in S or v is in S.
- Problem(Minimum vertex cover): Given a graph $G = (V, E)$, output the size of the *vertex cover* of G of minimum size.



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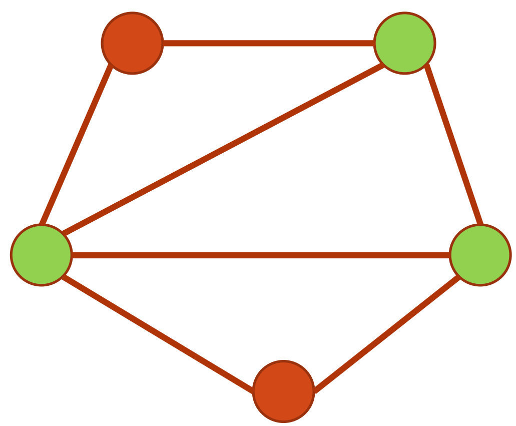

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- Problem(Minimum vertex cover): Given a graph $G = (V, E)$, output the size of the *vertex cover* of G of minimum size.

- Claim 1: Minimum-vertex-cover $\leq_p$ Vertex-cover.
- Claim 2: Vertex-cover $\leq_p$ Minimum-vertex-cover

Computational Intractability: Reduction

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  - Claim 3: G has an independent set of size at least k if and only if G has a vertex cover of size at most $(n - k)$.

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  - Claim 3: G has an independent set of size at least k if and only if G has a vertex cover of size at most $(n - k)$.
  - Given an instance of the independent set problem (G, k) create an instance of the vertex cover problem $(G, n - k)$, make a query to the black box solving the vertex cover problem and return the answer that is returned by the black box.

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  - Minimum-vertex-cover $\leq_p$ Vertex-cover.
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- <u>Proof 1</u>: G has an independent set of size at least k if and only if G has a vertex cover of size at most $(n - k)$.
- <u>Proof 2</u>:
  - Minimum-vertex-cover $\leq_p$ Vertex-cover.
  - Vertex-cover $\leq_p$ Independent-set.
  - Independent-set $\leq_p$ Maximum-independent-set.

- <u>Theorem</u>: If $X \leq_p Y$ and $Y \leq_p Z$, then $X \leq_p Z$.

Computational Intractability: Reduction

- Problem(Deg-3-Independent set): Given a graph $G = (V, E)$ of *bounded degree* 3 and an integer k, check if there is an *independent set* of size at least k in G.
  - *Graph with bounded degree 3*: A graph is said to have bounded degree 3 if the degrees of all vertices in the graph is at most 3.

- Claim: Independent-set $\leq_p$ Deg-3-Independent-set.

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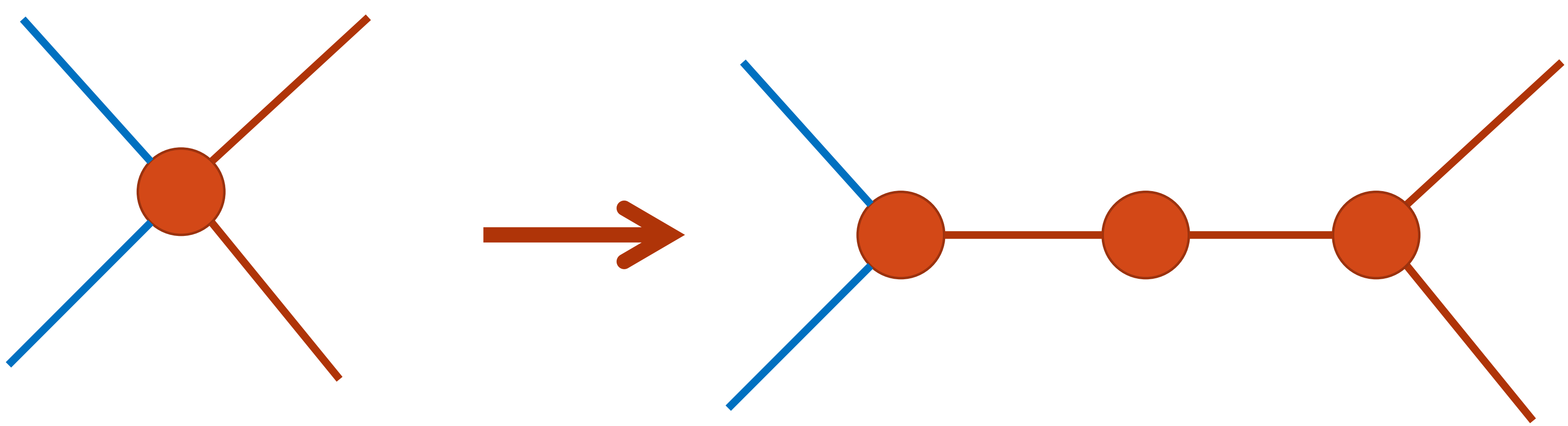
- Claim: Independent-set $\leq_p$ Deg-3-Independent-set.



Computational Intractability: Reduction

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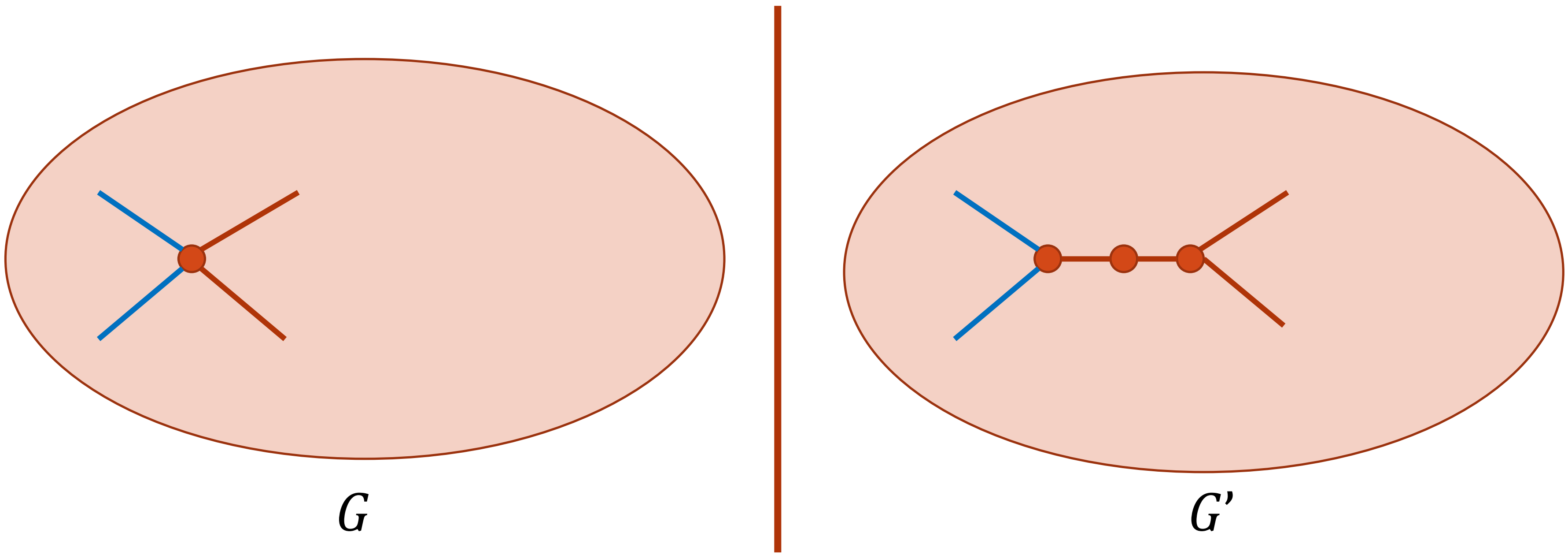

- Claim: G has an independent set of size at least k if and only if G' has an independent set of size at least $(k+1)$.

End