CSL 356: Analysis and Design of Algorithms

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Network Flow: Applications

Team elimination

Network Flow: Applications

- Problem: There are n teams. Each team i has a current number of wins denoted by $w(i)$. There are $G(i, j)$ games yet to be played between team i and j. For a given team x, has x been eliminated?

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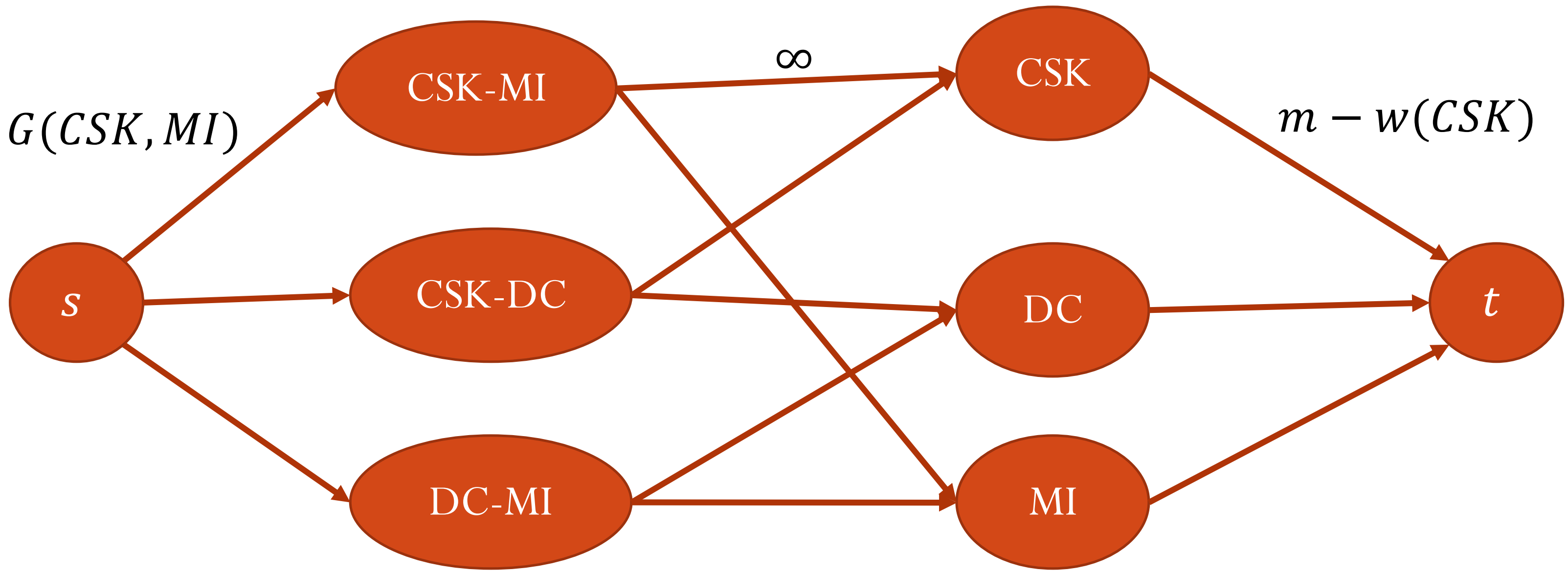
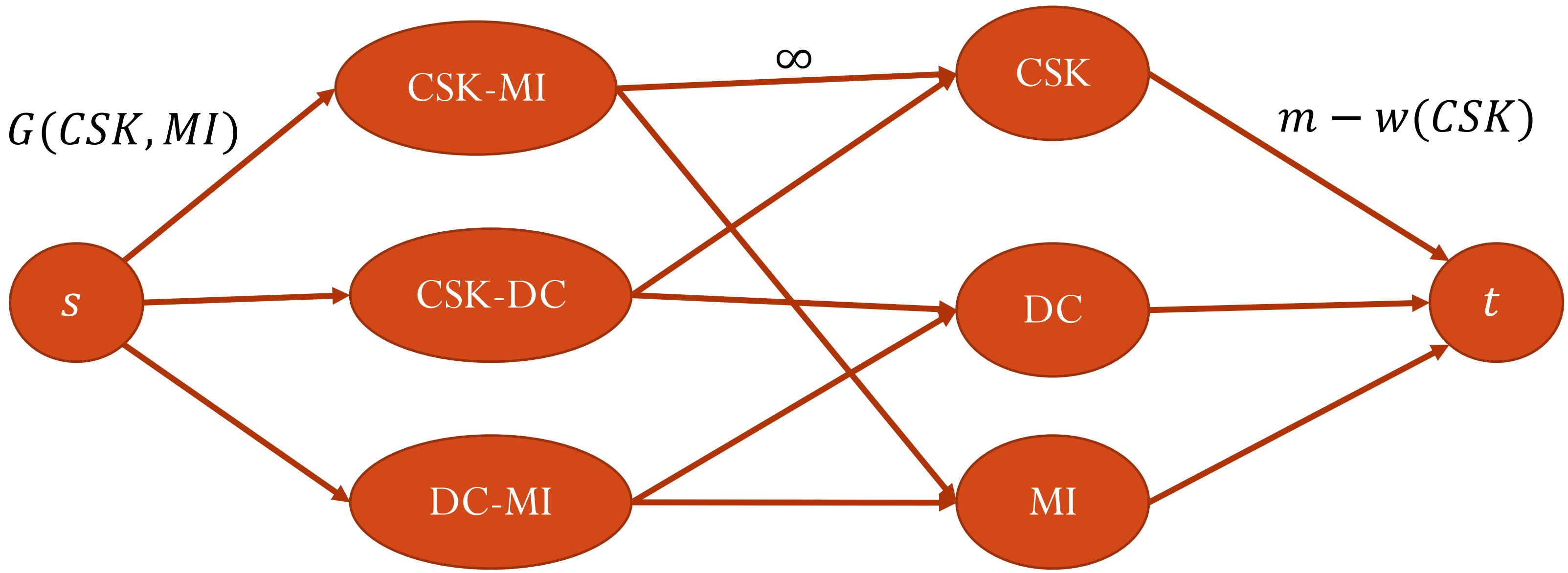


Team x can end with at most m wins, i.e., $m = w(x) + \sum_j G(x,j)$.

Network Flow: Applications

- Claim: Team x has been eliminated if and only if the maximum flow in the network is strictly less than $g^* = \sum_{i,j \neq x} G(i,j)$.



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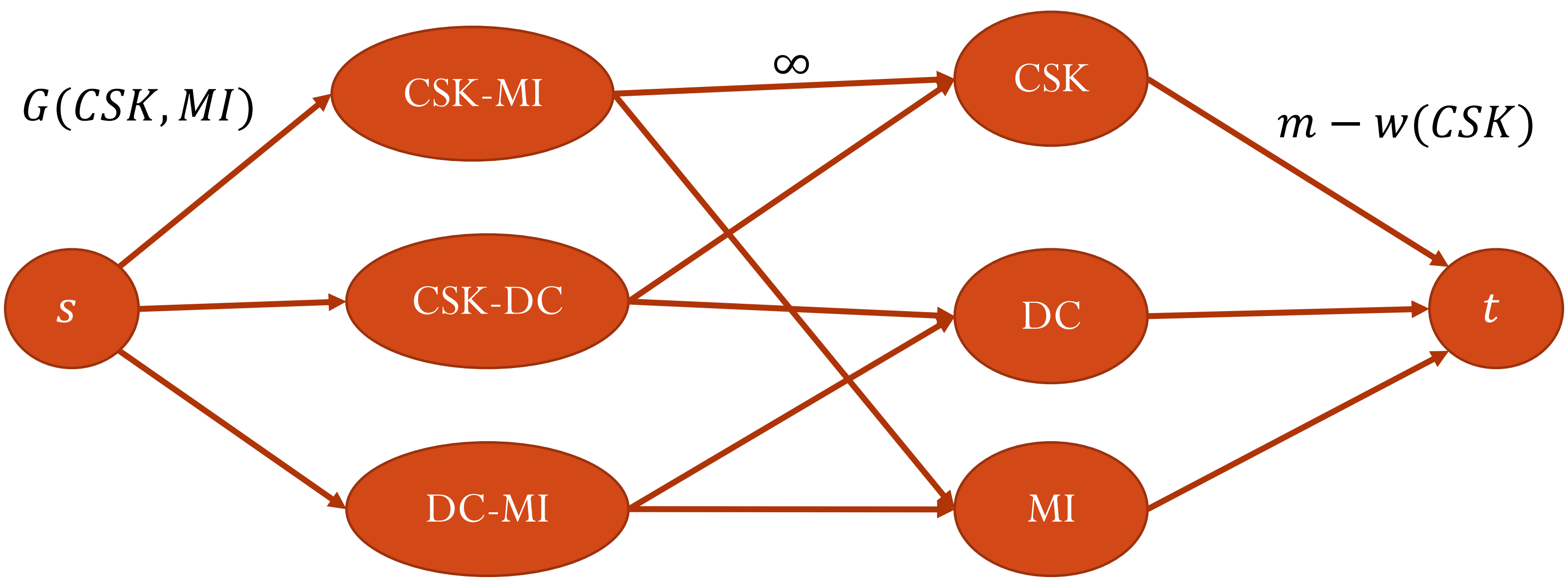
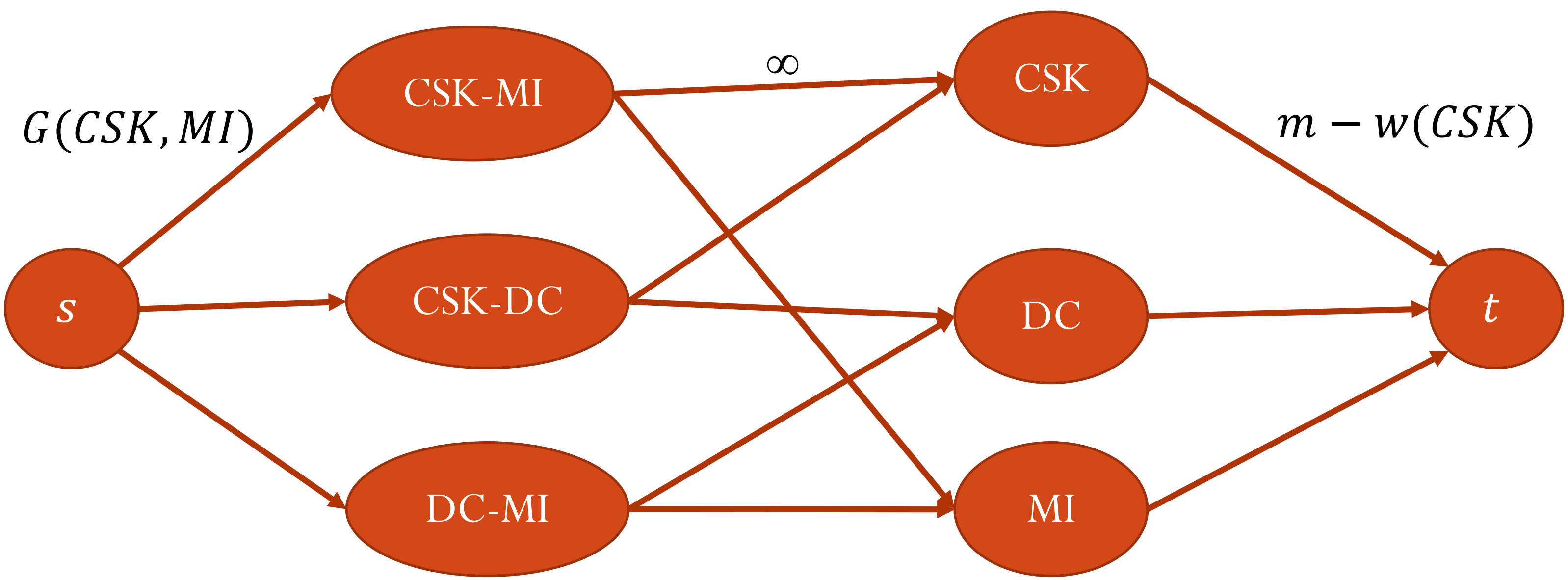
Network Flow: Applications

- If we can somehow find a subset T of teams (not including x) such that $\sum_{i \in T} w(i) + \sum_{i,j \in T, i<j} G(i,j) > m \cdot |T|$. Then we have a witness to the fact that x has been eliminated.
- Can we find such a subset T?



Network Flow: Applications

- (➔) Suppose x has been eliminated, then the max-flow in the network $< \ g^*$.
- (🡐) Consider any $s - t$ min-cut (A, B) in the graph.
  - Claim 1: If v_{ij} is in A, then both v_i and v_j are in A.



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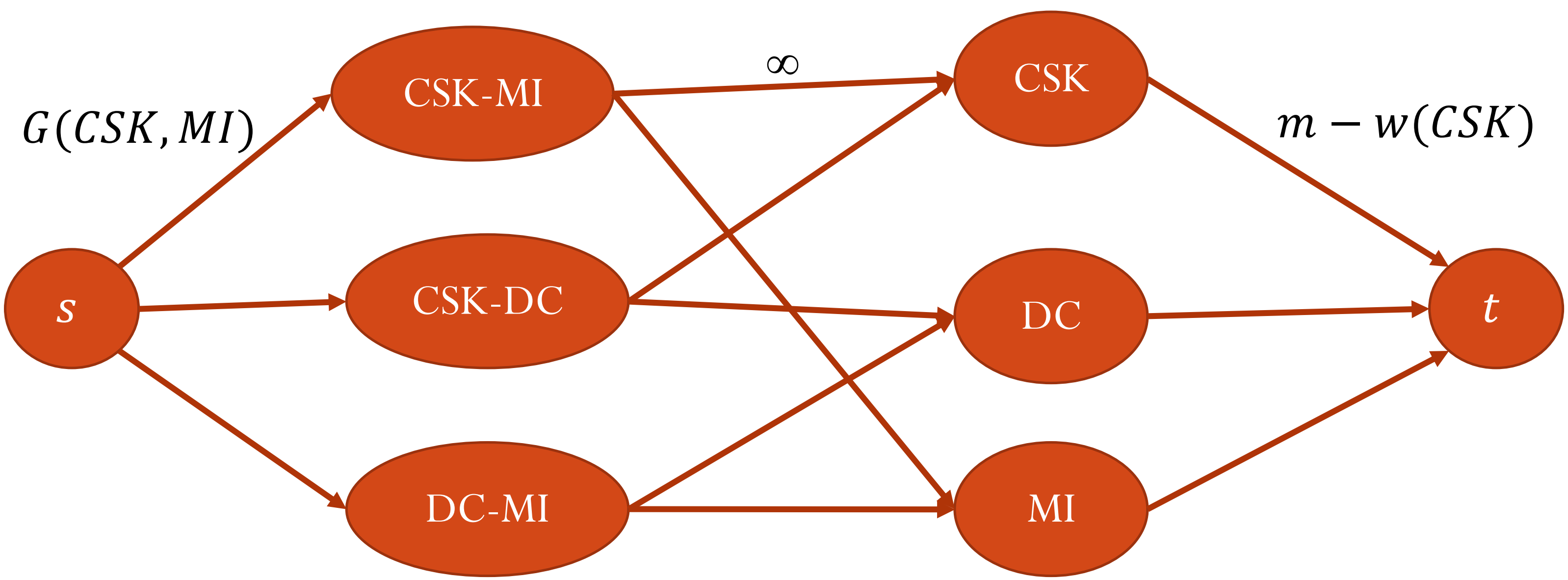

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$$C(A,B) = \sum_{i \in T}(m - w(i)) + \sum_{\{i,j\} \not\subset T} G(i,j) < g^*$$

$$\Rightarrow m \cdot |T| - \sum_{i \in T} w(i) + (g^* - \sum_{\{i,j\} \in T} G(i,j)) < g^*$$

$$\Rightarrow \sum_{i \in T} w(i) + \sum_{\{i,j\} \in T} G(i,j) > m \cdot |T|$$

Network Flow: Applications

Circulation with demands

Network Flow: Applications

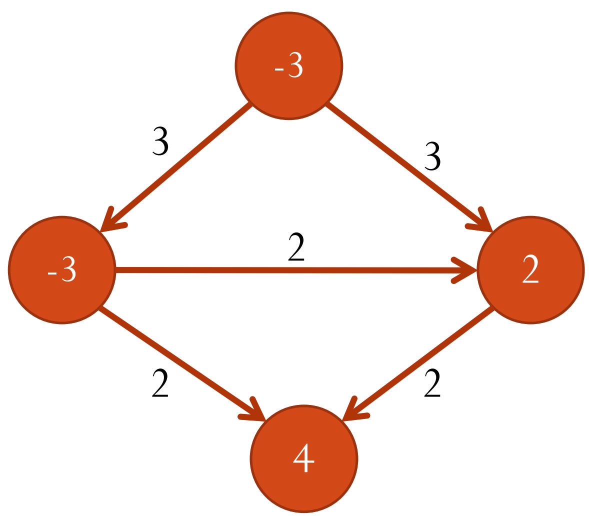
- Given a weighted directed graph representing a transportation network.
- There are multiple supply nodes in the graph denoting the places that has a factory for some product.
- There are multiple demand nodes denoting the consumption points.
- Each supply node v has an associated supply value $s(v)$ denoting the amount the product it can supply.
- Each demand node v has a similar demand value $d(v)$.
- <u>Question</u>: Is there a way to ship product such that all demand and supply goals are met?

Network Flow: Applications

- Problem: Given a directed graph G with integer edge capacities. For each node v, there is an associated demand value $t(v)$ denoting the demand at the node (for supply nodes this is $-s(v)$, for demand nodes $d(v)$, for other nodes 0). Find whether there exists a flow f such that for all nodes v:

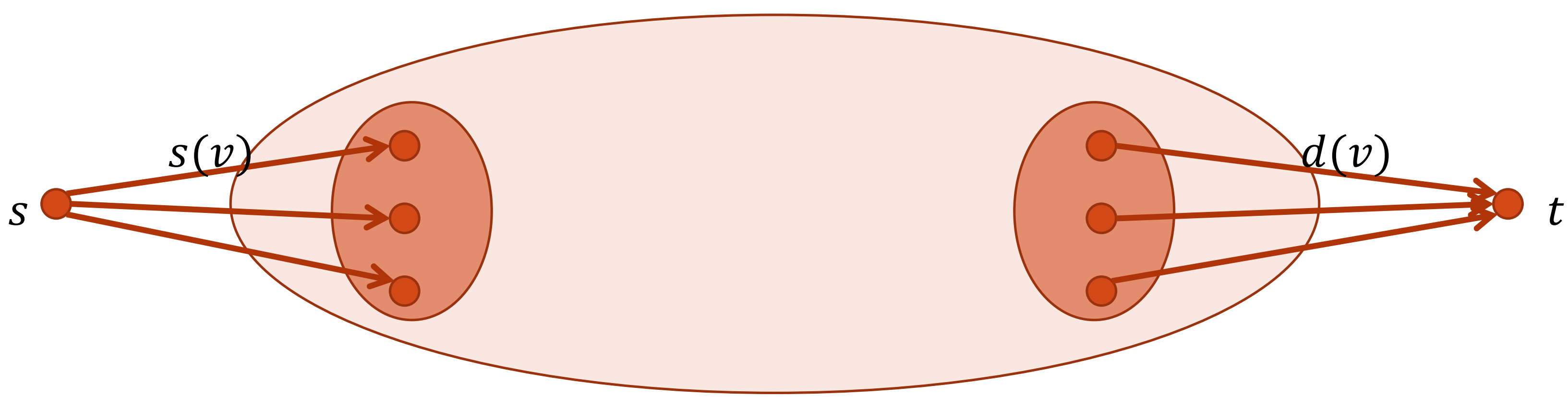
$$f^{in}(v) - f^{out}(v) = t(v)$$

  and the capacity constraints are met. Such a flow is called a *feasible circulation*.



Network Flow: Applications

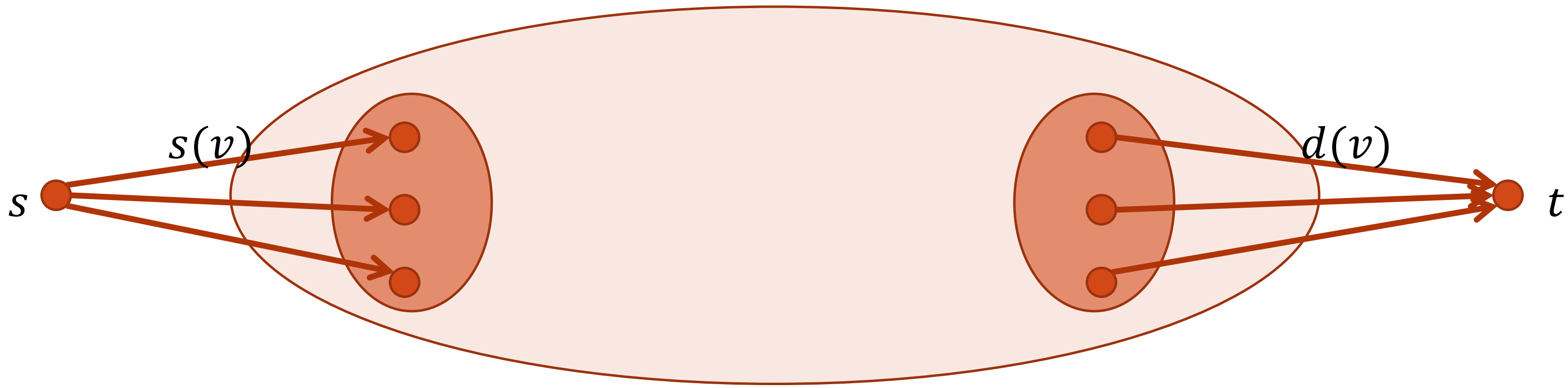
- Claim 1: For any feasible circulation f, $\sum_v t(v) = 0$. That means supply is equal to the demand.
- Consider the flow network and let $\mathrm{D} = \sum_{demand\ nodes\ v} d(v)$.
- Claim 2: There is a feasible circulation in G if and only if the maximum $s - t$ flow in the flow network is D.



Network Flow: Applications

- Claim 2: There is a feasible circulation in G if and only if the maximum $s - t$ flow in the flow network is D.
  - Proof:

    (➔)



Network Flow: Applications

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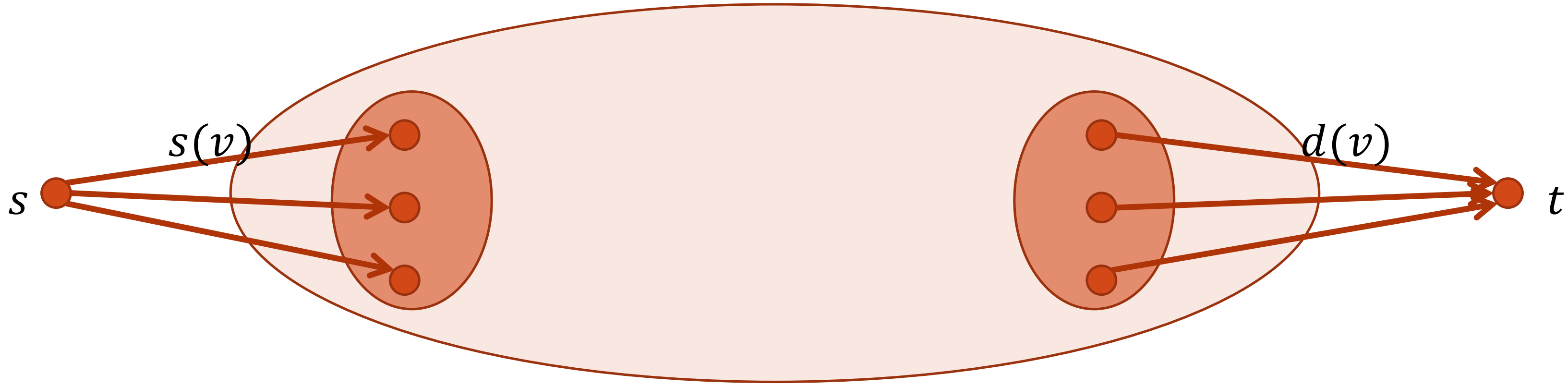
    (➔) extend the feasible circulation for the network.



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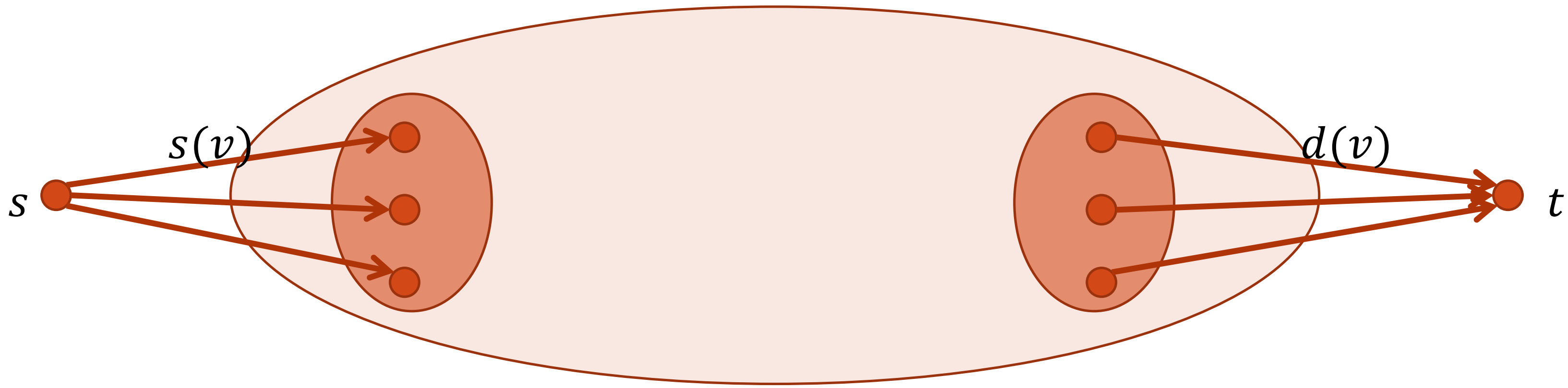
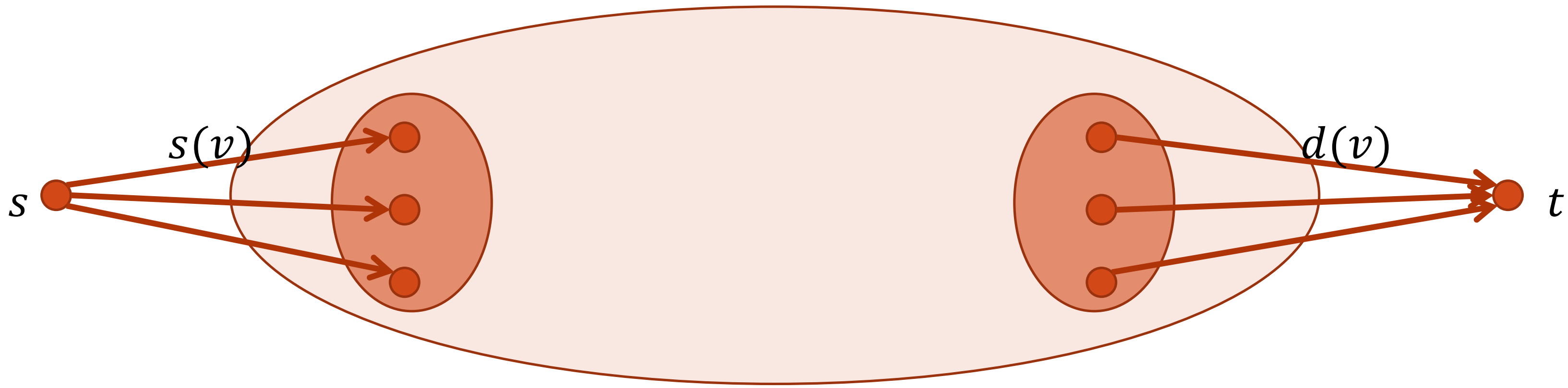


Network Flow: Applications

- Claim 2: There is a feasible circulation in G if and only if the maximum $s - t$ flow in the flow network is D.
  - Proof:
    (➔) extend the feasible circulation for the network.
    (🡐) Consider max $s - t$ flow and just remove s and t.



Network Flow: Applications

Supply and Demand with capacity

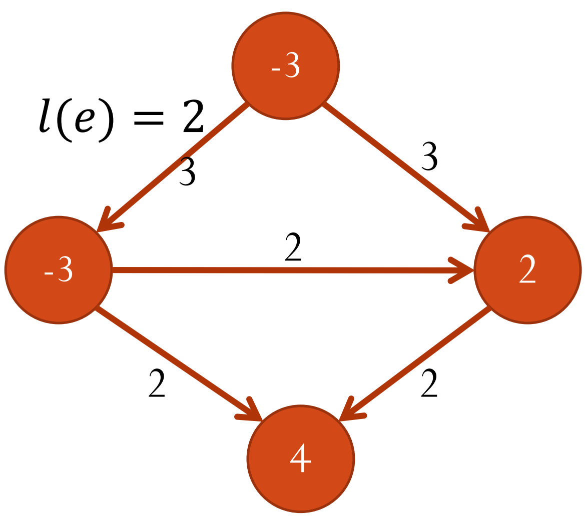
Network Flow: Applications

- Problem: Given a directed graph G with integer edge capacities $c(e)$ and a lower bound $l(e)$. For each node v, there is an associated demand value $t(v)$ denoting the demand at the node (for supply nodes this is $-s(v)$, for demand nodes $d(v)$, for other nodes 0). Find whether there exists a flow f such that for all nodes v:

$$f^{in}(v) - f^{out}(v) = t(v)$$

and the following capacity constraints are met. For each edge e:

$$l(e) \leq f(e) \leq c(e)$$



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