CSL 356: Analysis and Design of Algorithms

Ragesh Jaiswal

CSE, IIT Delhi

Network Flow: Applications

Hall's Theorem

Network Flow: Applications

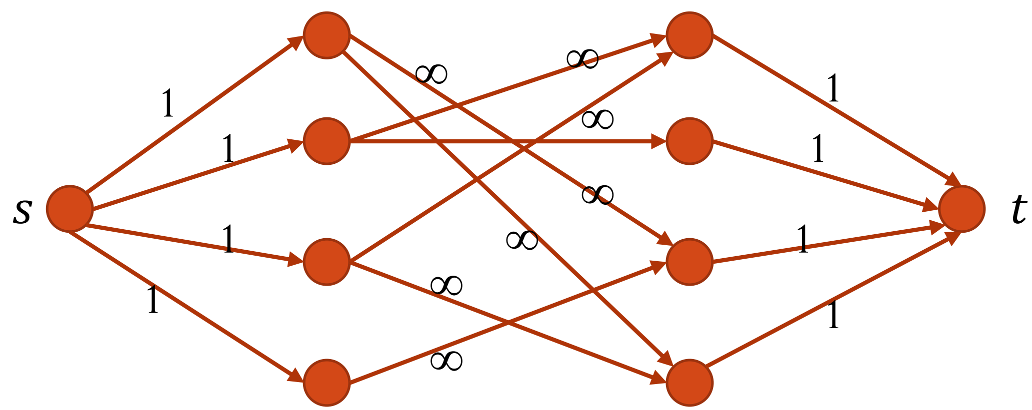
- Hall's Theorem: Given any bipartite graph $G = (X, Y, E)$, there is a perfect matching in G if and only if for every subset A of vertices of X, we have $|A| \leq |N(A)|$.

- Claim 1(➔): If there is a perfect matching then for all subsets A of X, $|A| \leq |N(A)|$.

Network Flow: Applications

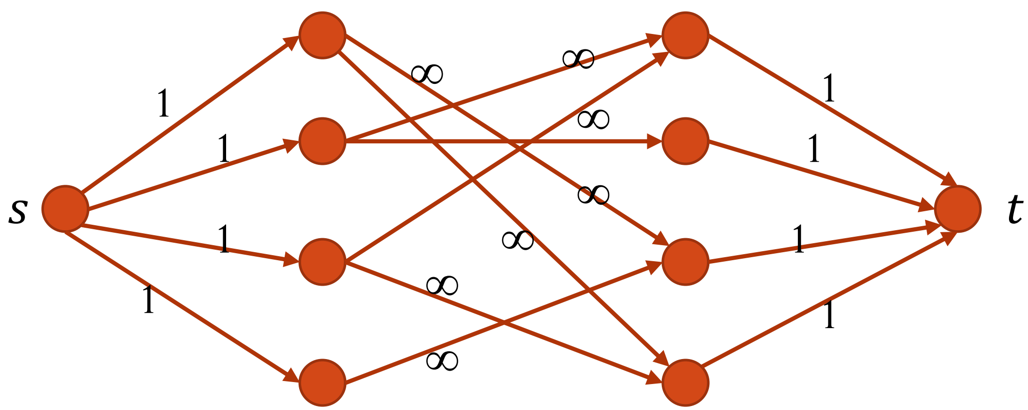
- Hall's Theorem: Given any bipartite graph $G = (X, Y, E)$, there is a perfect matching in G if and only if for every subset A of vertices of X, we have $|A| \leq |N(A)|$.

- Claim 1(➔): If there is a perfect matching then for all subsets A of X, $|A| \leq |N(A)|$.
- Claim 2(🡐): If there is no perfect matching then there is a subset A of X such that $|A| > |N(A)|$.
  - Proof: Consider the following flow network.



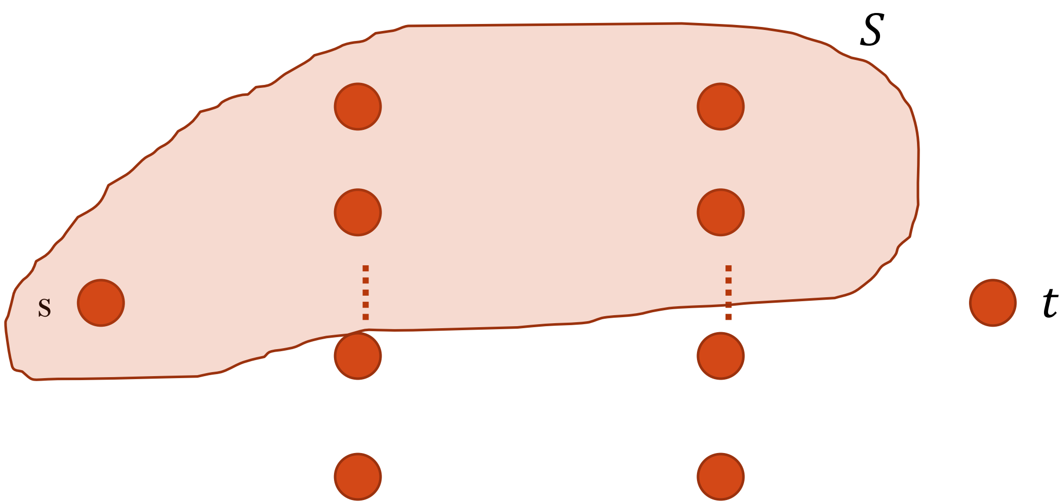
Network Flow: Applications

- Claim 2(🡐): If there is no perfect matching then there is a subset A of X such that $|A| > |N(A)|$.
  - Proof: Consider the flow network.
    - Claim 2.1: The max-flow in the network is equal to the maximum matching in G.



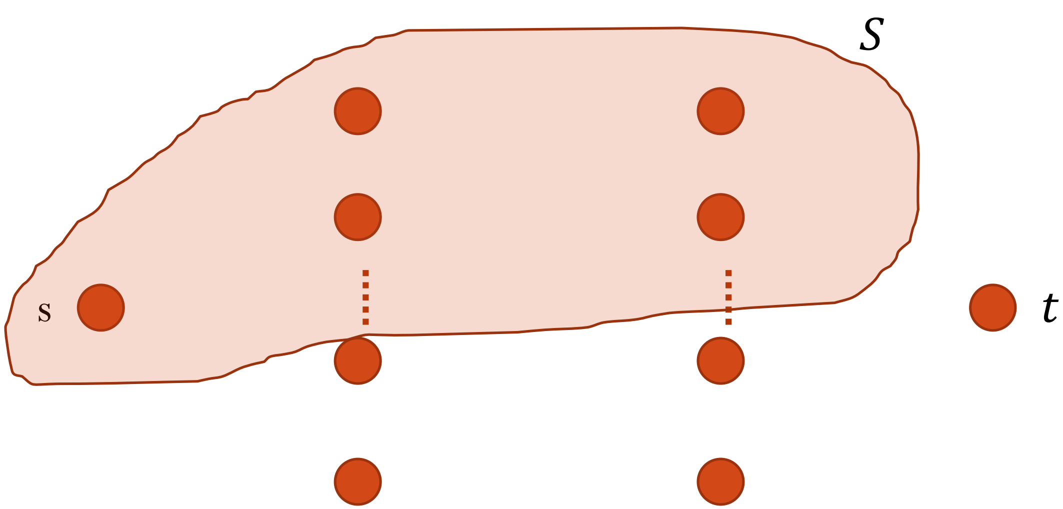
Network Flow: Applications

- Claim 2(🡐): If there is no perfect matching then there is a subset A of X such that $|A| > |N(A)|$.
  - Proof: Consider the flow network.
    - Claim 2.1: The max-flow in the network is equal to the maximum matching in G.
    - Let f be the max integer flow in the network. Consider the residual graph G_f. Let S be the set of all vertices reachable from s in G_f. Let A' be vertices of X in S and B' be vertices of Y in S.



Network Flow: Applications

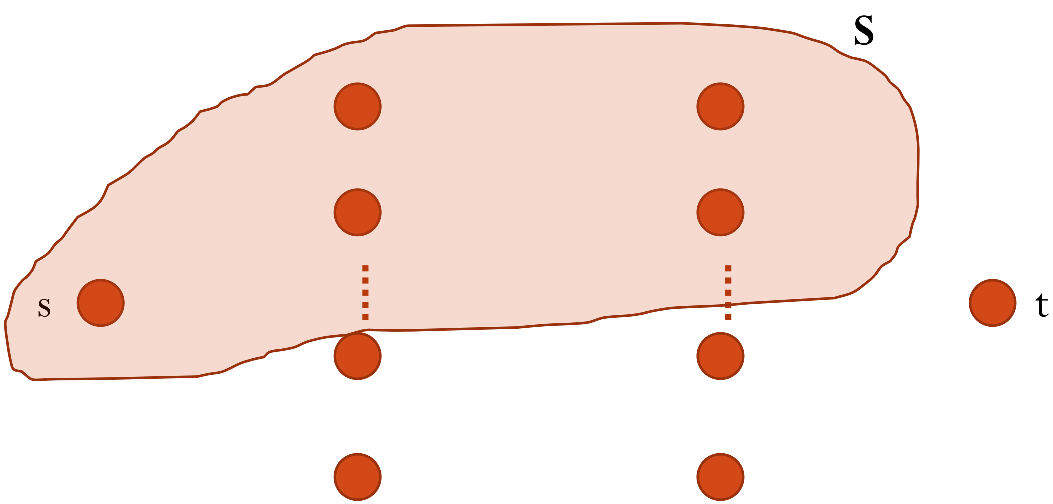
- Claim 2(🡐): If there is no perfect matching then there is a subset A of X such that $|A| > |N(A)|$.
  - Proof: Consider the flow network.
    - Let f be the max integer flow in the network. Consider the residual graph G_f. Let **S** be the set of all vertices reachable from s in G_f. Let A' be vertices of X in S and B' be vertices of Y in S.
    - Claim 2.2: $B' = N(A')$.



Network Flow: Applications

- Claim 2(🡐): If there is no perfect matching then there is a subset A of X such that $|A| > |N(A)|$.
  - Proof: Consider the flow network.
    - Let f be the max integer flow in the network. Consider the residual graph G_f. Let S be the set of all vertices reachable from s in G_f. Let A' be vertices of X in S and B' be vertices of Y in S.
    - Claim 2.2: $B' = N(A')$.
    - Capacity of the cut = max-flow $< n = |X|$

$$\Rightarrow n - |A'| + |N(A')| < n$$
$$\Rightarrow |A'| > |N(A')|$$



Network Flow: Applications

- Hall's Theorem: Given any bipartite graph $G = (X, Y, E)$, there is a perfect matching in G if and only if for every subset A of vertices of X, we have $|A| \leq |N(A)|$.

- If there is no perfect matching, the maximum flow in the network also gives a subset A for which $|A| > |N(A)|$.
- This can be considered a *certificate* of the fact that there is no perfect matching in G.

Network Flow: Applications

Team elimination

Network Flow: Applications

- Suppose there are 4 teams in the IPL with their current number of wins:
  - CSK: 10
  - MI: 10
  - DC: 10
  - KKR: 8
- There are 7 more games to be played.
  - KKR plays all 3 teams
  - CSK vs MI, MI vs DC, DC vs CSK, and MI vs CSK.
- A team is said to be eliminated if it cannot end with maximum number of wins.
- Can we say that KKR has been eliminated?

Network Flow: Applications

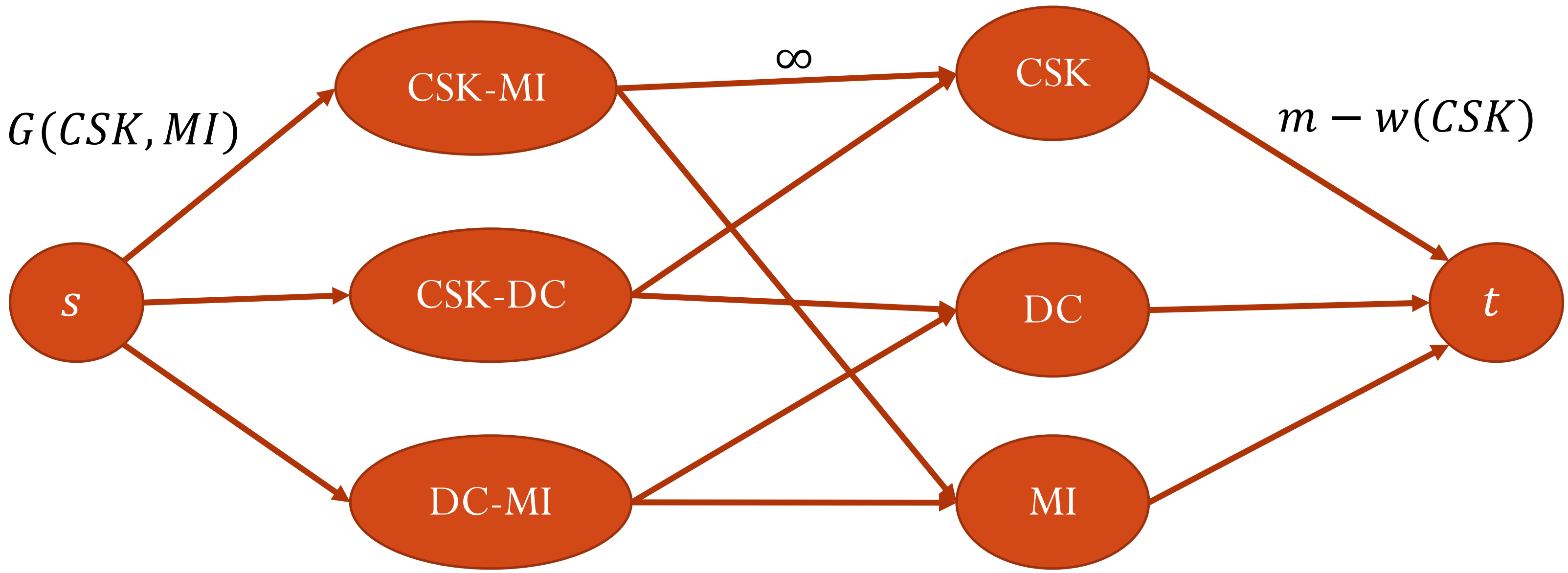
- Suppose there are 4 teams in the IPL with their current number of wins:
  - CSK: 10
  - MI: 10
  - DC: 9
  - KKR: 8
- There are 7 more games to be played.
  - KKR plays all 3 teams
  - CSK vs MI, CSK vs MI, CSK vs MI, CSK vs MI.
- Can we say that KKR has been eliminated?

Network Flow: Applications

- Problem: There are n teams. Each team i has a current number of wins denoted by $w(i)$. There are $G(i, j)$ games yet to be played between team i and j. For a given team x, has x been eliminated?

Network Flow: Applications

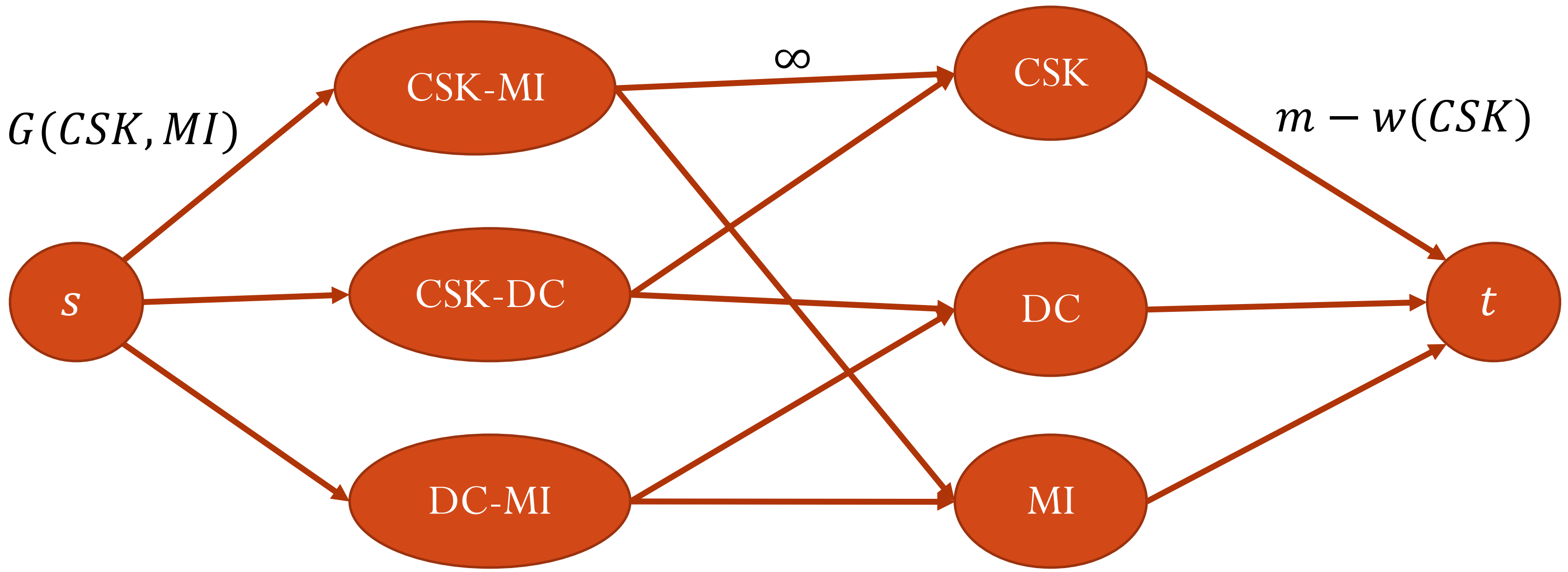
- Problem: There are n teams. Each team i has a current number of wins denoted by $w(i)$. There are $G(i,j)$ games yet to be played between team i and j. For a given team x, has x been eliminated?



Team x can end with at most m wins, i.e., $m = w(x) + \sum_j G(x,j)$.

Network Flow: Applications

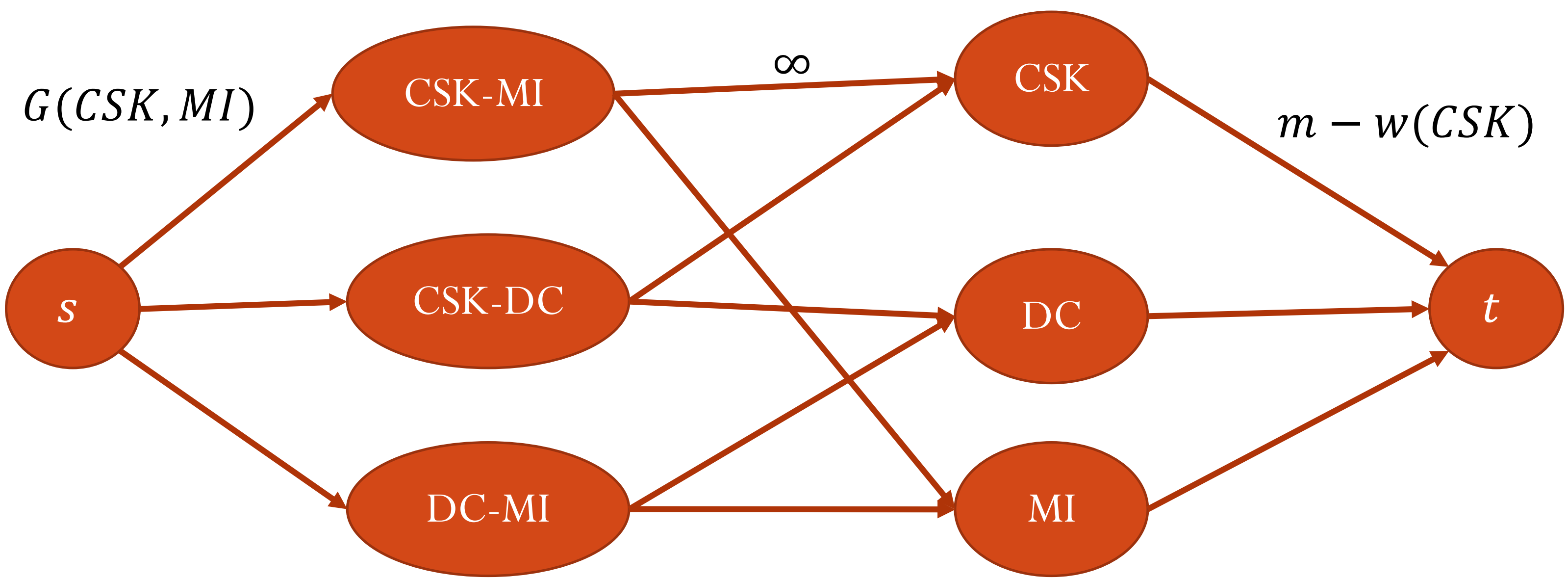
- <u>Claim</u>: Team x has been eliminated if and only if the maximum flow in the network is strictly less than $g^* = \sum_{i,j \neq x} G(i,j)$.



Team x can end with at most m wins, i.e., $m = w(x) + \sum_j G(x,j)$.

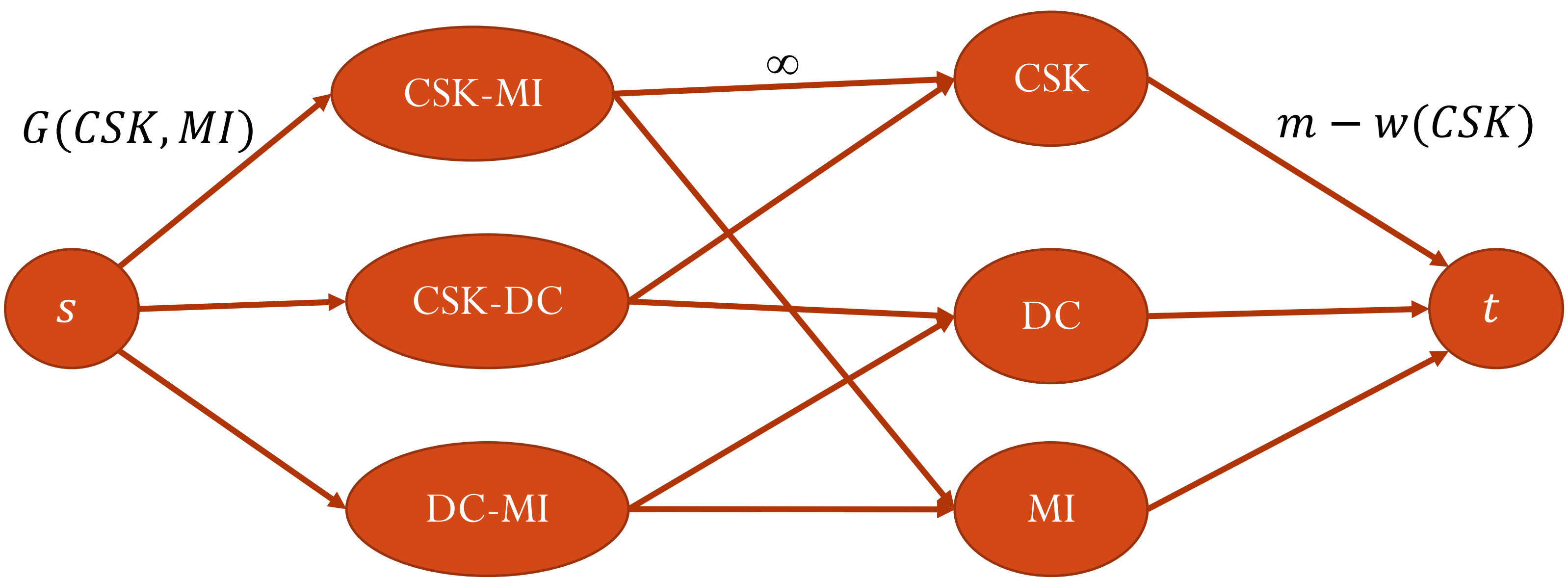
Network Flow: Applications

- If we can somehow find a subset T of teams (not including x) such that $\sum_{i \in T} w(i) + \sum_{i,j \in T, i<j} G(i,j) > m \cdot |T|$. Then we have a witness to the fact that x has been eliminated.
- Can we find such a subset T?



Network Flow: Applications

- (➔) Suppose x has been eliminated, then the max-flow in the network $< g^*$.



End