CSL 356: Analysis and Design of Algorithms

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Divide and Conquer: Examples

Median Finding: Finding the k^{th} smallest number in an unsorted array.

Divide and Conquer: Examples

- Problem(Median Finding): Given an array A of unsorted numbers and an integer k. Give an algorithm that finds the k^{th} smallest number in the array. Assume A contains distinct numbers.
- Divide and Conquer:
  - Pick an number p as pivot. Partition the numbers in A into A_L (all numbers < p) and A_R (all numbers > p).
    - If $|A_L| = k-1$, then output p.
    - If $|A_L| > k-1$, then recursively find the k^{th} smallest number in A_L
    - If $|A_L| < k-1$, then recursively find the $(k-|A_L|-1)^{th}$ smallest number in A_R.

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- What is the running time of this algorithm?

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- What is the running time of this algorithm?
  - If we pick a bad pivot each time, then the running time can be as bad as $O(n^2)$.

Divide and Conquer: Examples

- How do we pick a good pivot number?
  - Randomly: We will look at this a bit later.
  - Deterministically: ?

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- Consider groups of 5 elements
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- Consider the median of the medians:
  - Here it is 46.
- Use this as the pivot element.

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- How many elements in A are larger than p?

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  - <u>Claim</u>: There are at least $(3n/10 - 6)$ numbers in A that are larger than p.

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- Consider the median of the medians:
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- Use this as the pivot element **p**.

- How many elements in A are smaller than p?
  - <u>Claim</u>: There are at least $(3n/10 - 6)$ numbers in A that are smaller than p.

Divide and Conquer: Examples

Find-kth-smallest(A, k)

- ... *//Base cases*
- Consider groups of 5 numbers, sort each group and create another array B containing the median numbers from each group.
- p ← Find-kth-smallest(B, floor($|B|/2$))
- Partition the array A into A_L and A_R using p as the pivot.
- If ($|A_L| = k - 1$) then output(p)
- If ($|A_L| > k - 1$) then output(Find-kth-smallest(A_L, k))
- If ($|A_L| < k - 1$) then output(Find-kth-smallest($A_R, k - |A_L| - 1$))

- Running time:?

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- Running time:
  - $T(n) = T(\lceil n/5 \rceil) + T(7n/10 + 6) + O(n)$
  - $T(n) = ?$

Divide and Conquer: Examples

- Suppose we want to find the 12th smallest element in the array?

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- Make a recursive call to find median element of medians.

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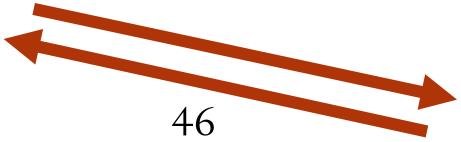

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- Partition using the pivot as 46.

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11 elements

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Divide and Conquer: Examples

Fast Fourier Transform (FFT)

Divide and Conquer: Examples

- Problem: Given two polynomials
  $A(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_{n-1} \cdot x^{n-1}$,
  and
  $B(x) = b_0 + b_1 \cdot x + b_2 \cdot x^2 + \ldots + b_{n-1} \cdot x^{n-1}$
  multiply them.

- We have to obtain the polynomial $C(x) = A(x) \cdot B(x)$
  $C(x) = c_0 + c_1 \cdot x + c_2 \cdot x^2 + \ldots + c_{2n-2} * \cdot x^{2n-2}$

- What is c_i in terms of coefficients of A and B?

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- What is c_i in terms of coefficients of A and B?
  - $c_i = a_i \cdot b_0 + a_{i-1} \cdot b_1 + a_{i-2} \cdot b_2 + \ldots + a_0 \cdot b_i$

- The vector $(c_0, \ldots, c_{2n-2})$ is called the *convolution* of vectors $(a_0, \ldots, a_{n-1})$ and $(b_0, \ldots, b_{n-1})$.

Divide and Conquer: Examples

SimpleMultiply($(a_0, \ldots, a_{n-1}), (b_0, \ldots, b_{n-1})$)

- For $i = 0$ to $2n - 2$
  - For $j = 0$ to i
    - $c_i = c_i + a_j \cdot b_{i-j}$
- return($(c_0, \ldots, c_{2n-2})$)

- What is the running time of the simple algorithm?
  - $O(n^2)$
- Is there another way to compute the polynomial $C(x)$?

Divide and Conquer: Examples

```
SimpleMultiply((a_0, …, a_{n−1}), (b_0, …, b_{n−1}))
    - For i = 0 to 2n − 2
        - For j = 0 to i
            - c_i = c_i + a_j · b_{i−j}
    - return((c_0, …, c_{2n−2}))
```

- What is the running time of the simple algorithm?
  - $O(n^2)$
- Is there another way to compute the polynomial $C(x)$?
  - Compute $A(s_1), A(s_2), \dots, A(s_{2n})$.
  - Compute $B(s_1), B(s_2), \dots, B(s_{2n})$.
  - Compute
    - $C(s_1) = A(s_1) \cdot B(s_1)$,
    - …
    - $C(s_{2n}) = A(s_{2n}) \cdot B(s_{2n})$.

Divide and Conquer: Examples

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    - $C(s_{2n}) = A(s_{2n}) \cdot B(s_{2n})$.
  - *Interpolate* to obtain the polynomial $C(x)$.
- How fast can you compute $A(s)$ for a given value of s?

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  - *Interpolate* to obtain the polynomial $C(x)$.
- How fast can you compute $A(s)$ for a given value of s?
  - $O(n)$ arithmetic operations using *Horner's rule*.
  - $A(s) = a_0 + s \cdot (a_1 + s \cdot (a_2 + \cdots + s \cdot (a_{n-2} + s \cdot (a_{n-1})) \dots))$

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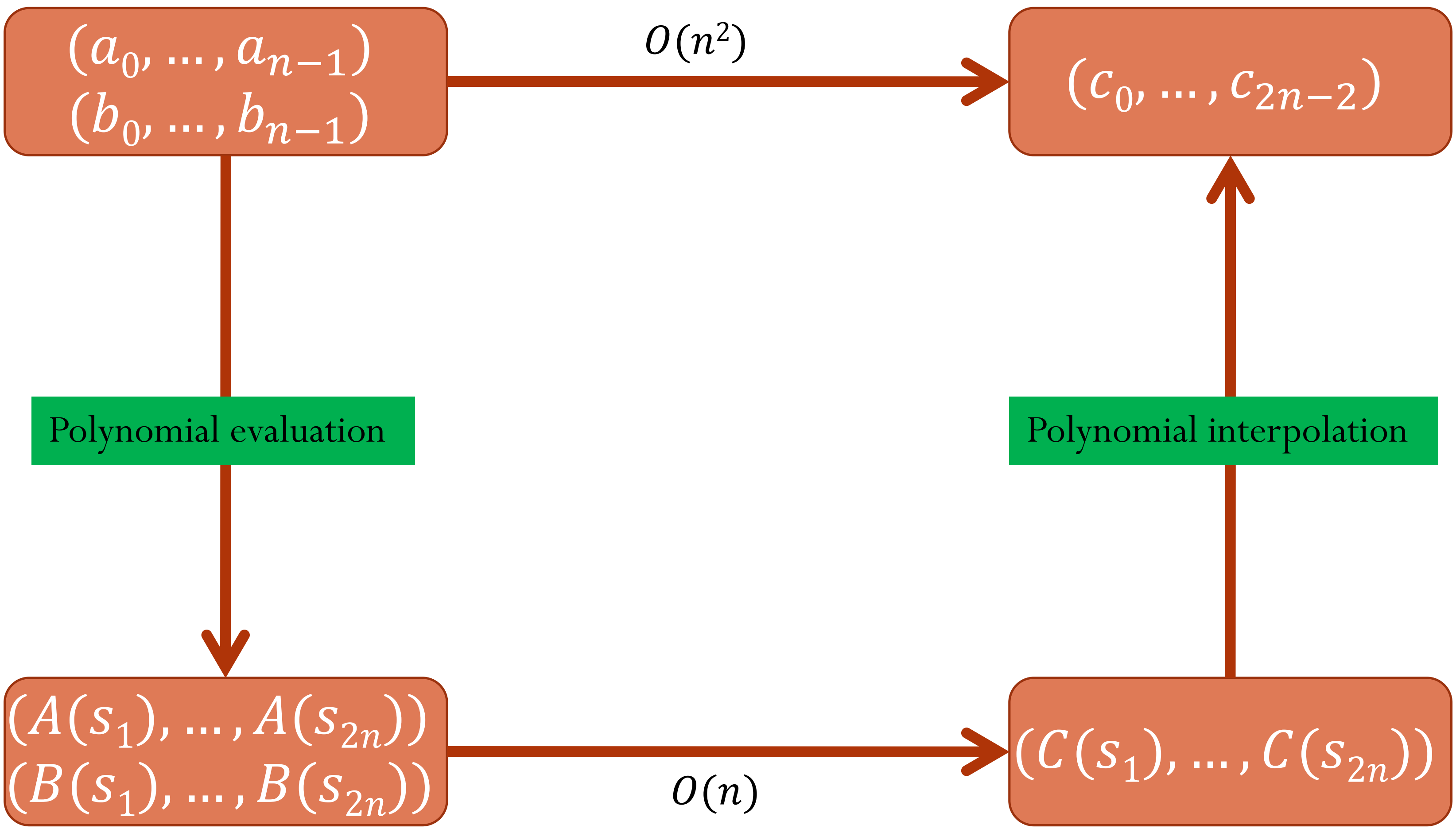

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