- **Use of unfair means will be severely penalized.**

There are 4 questions for a total of 50 points.

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(15) 1. You are given a bipartite graph $G = (X, Y, E)$ such that $|X| = |Y|$. Furthermore, every vertex in G has degree exactly k. Show that the edges of the graph can be colored with k distinct colors so that no two edges incident at a vertex have the same color.

(10) 2. Prove or disprove: Among $n + 2$ arbitrarily chosen integers, either there are two whose difference is divisible by $2n$ or there are two whose sum is divisible by $2n$.

(10) 3. Show the following version of the birthday lemma:

Let N and r be positive integers and let S be a set of size N. Suppose we pick r elements $Y_1, ..., Y_r$ from the set S randomly with replacement and then pick another r elements $Z_1, ..., Z_r$ randomly with replacement from S. Let $D(N, r)$ denote the probability that there is a pair (i, j) such that $Y_i = Z_j$. Show that $D(N, r) \geq C(n, 2r)/2$.

(Recall, $C(N, 2r)$ is the probability that $2r$ randomly chosen elements from S are not all distinct.)

(15) 4. We say that a string of bits has k triply-repeated ones if there are k positions where three consecutive 1s appear in a row. For example, the string 011100111110 has four triply-repeated ones. Consider an experiment that outputs a random n-bit string (i.e., all n bit strings are equally likely). Let X be a random variable denoting the number of triply repeated 1s in the n-bit string. What is the value of $\sum_{i=0}^{n-2} i \cdot \mathbf{Pr}[X = i]$?