There are 5 questions for a total of 50 points.

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(10) 1. A cat and a mouse each independently take a random walk on a connected, undirected, non-bipartite graph $G = (V, E)$. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Show an upper bound of $O(|E|^2 \cdot |V|)$ on the expected time before the cat eats the mouse.

(10) 2. In a connected graph G, an edge is called a bridge if the removal of the edge disconnects the graph. Let G be a connected graph with n vertices and m edges. Let (u, v) be any edge in G. For the simple random walk on G, show that $h_{uv} + h_{vu} = 2m$ if and only if the edge (u, v) is a bridge.

(10) 3. Exercise 10.6 from the chapter on "The Monte Carlo Method" (Chapter 10).

(10) 4. Let n, k be positive integers with $k \leq n/2$, and let Ω denote the set of all subsets of $\{1, ..., n\}$ of cardinality k. Consider the following Markov Chain on state space Ω: from a subset S, pick an element $a \in S$ and an element $b \in \{1, ..., n\} \setminus S$, independently and uniformly at random, and make a transition to the set $(S - \{a\}) \cup \{b\}$. Make a minor change in the Markov Chain to construct a Chain that is ergodic. Analyse the mixing time of your Markov Chain using a coupling argument.

(10) 5. Recall card shuffling discussed in class. Consider the following new way of shuffling a deck of n cards:

> Pick card c and a position p uniformly at random and then exchange card c with the card at position p in the deck.

Consider the following coupling $Z_t = (X_t, Y_t)$ for analyzing the mixing time of this Markov Chain: X_t and Y_t chooses the same c and p at each step. Use this coupling to show that the mixing time is $O(n^2)$.