Hash Functions

CSL 866: Cryptography

IIT Delhi

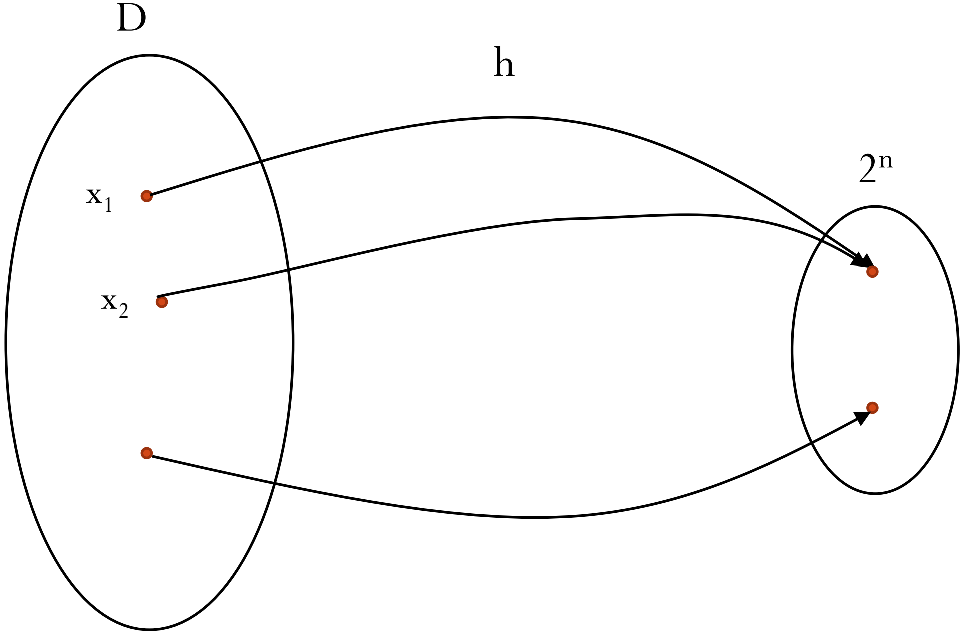
What is a hash function?

By a hash function we usually mean a map $h : D \to \{0,1\}^n$ that is compressing, meaning $|D| > 2^n$.

E.g. $D = \{0,1\}^{\leq 2^{64}}$ is the set of all strings of length at most 2^{64}.

h	n
MD4	128
MD5	128
SHA1	160
RIPEMD	128
RIPEMD-160	160
SHA-256	256
Skein	256, 512, 1024

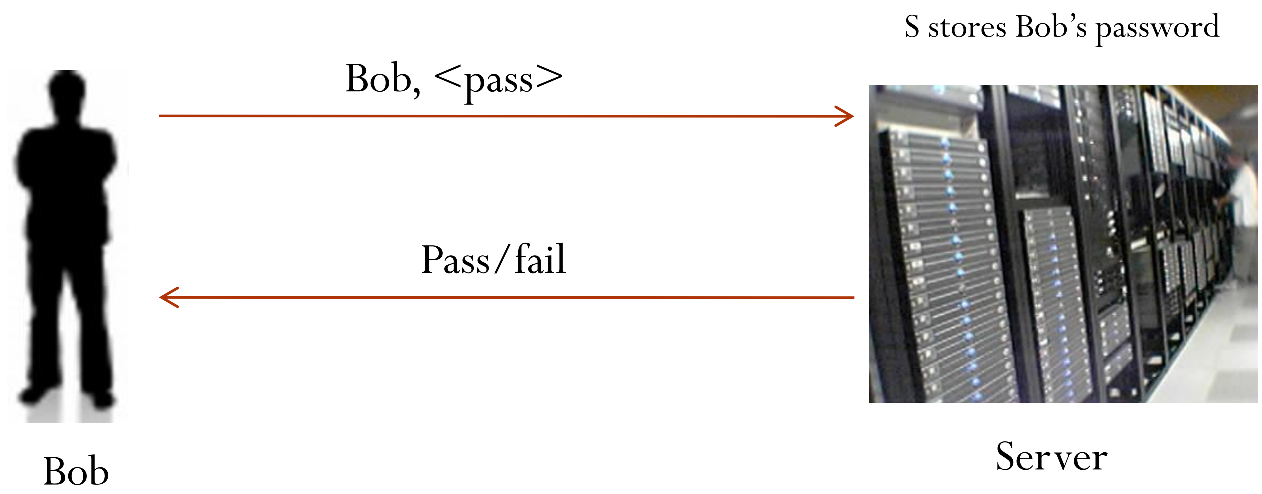
Hash Functions: Collision



Pigeonhole Principle: $h(x_1) = h(x_2)$, $x_1 \neq x_2$

Hash Functions: Applications

1. Password Authentication:



- <u>Problem</u>: If Eve hacks into the server or if the communication channel is not secure, then Eve knows the password of Bob.

Hash Functions: Applications

1. Password Authentication:

S stores h(Bob’s password)

Bob, h(<pass>)

Pass/fail

Bob

Server

- Eve can only get access to h(<pass>).

Hash Functions: Applications

1. Comparing files by hashing:

S has F_A

Server A

F_A →

← Yes/No

S has F_B

Server B

- <u>Problem</u>: Files are usually very large and we would like to save communication costs/delays.

Hash Functions: Applications

1. Comparing files by hashing:

S has F_A

Server A

$h(F_A)$ →

← $h(F_A) = h(F_B)$

S has F_B

Server B

Collision Resistance

- Password Authentication: If Eve is able to find a string S (even different from <pass>) such that

$$h(S) = h(<pass>)$$

then the scheme breaks.

- Comparing files: If there is a different file F_S such that

$$h(F_S) = h(F_B)$$

the servers may agree incorrectly.

- Collision Resistance: It is computationally infeasible to find a pair (x_1, x_2) such that $x_1 \neq x_2$ and

$$h(x1) = h(x2)$$

- If a hash function h is collision resistant, then the above two problems are avoided.

Collision Resistance: Discussion

- Are there functions that are collision resistant?
  - Fortunately, there are functions for which no one has been able to find a collision!
  - Example: SHA-1: $\{0,1\}^D \longrightarrow \{0,1\}^{160}$
- Is the world drastically going to change if someone finds one or few collision for SHA-1?
  - Not really. Suppose the collision has some very specific structure, then we may avoid such structures in the strings on which the hash function is applied.
  - On the other hand, if no one finds a collision then that is a very strong notion of security and we may sleep peacefully without worrying about maintaining complicated structures in the strings.
  - We are once again going for a very strong definition of security for our new primitive similar to block ciphers and SE.

Function families

We consider a family $H : \{0,1\}^k \times D \to \{0,1\}^n$ of functions, meaning for each K we have a map $h = H_K : D \to \{0,1\}^n$ defined by

$$h(x) = H(K, x)$$

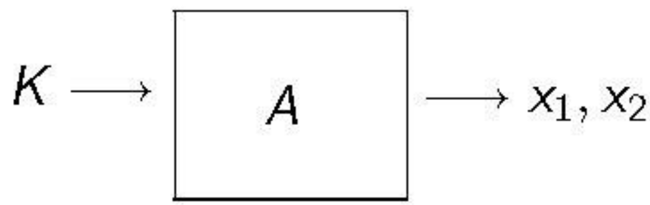
Usage: $K \xleftarrow{\$} \{0,1\}^k$ is made public, defining hash function $h = H_K$.

Note the key K is not secret. Both users and adversaries get it.

CR of function families

Let $H : \{0,1\}^k \times D \to \{0,1\}^n$ be a family of functions. A cr-adversary A for H

- Takes input a key $K \in \{0,1\}^k$
- Outputs a pair $x_1, x_2 \in D$ of points in the domain of H



A wins if x_1, x_2 are a collision for H_K, meaning

- $x_1 \neq x_2$, and
- $H_K(x_1) = H_K(x_2)$

Denote by $\mathbf{Adv}_H^{\mathrm{cr}}(A)$ the probability that A wins.

CR of function families

Let $H : \{0,1\}^k \times D \to \{0,1\}^n$ be a family of functions and A a cr-adversary for H.

Game CR_H

procedure Initialize	procedure Finalize(x_1, x_2)
$K \xleftarrow{\$} \{0,1\}^k$	Return $(x_1 \neq x_2 \wedge H_K(x_1) = H_K(x_2))$
Return K	

Let

$$\mathbf{Adv}_H^{\mathrm{cr}}(A) = \Pr\left[\mathrm{CR}_H^A \Rightarrow \mathsf{true}\right].$$

The measure of success

Let $H\colon \{0,1\}^k \times D \to \{0,1\}^n$ be a family of functions and A a cr adversary. Then

$$\mathbf{Adv}_H^{\mathrm{cr}}(A) = \Pr\left[\mathrm{CR}_H^A \Rightarrow \mathsf{true}\right].$$

is a number between 0 and 1.

A "large" (close to 1) advantage means

- A is doing well
- H is not secure

A "small" (close to 0) advantage means

- A is doing poorly
- H resists the attack A is mounting

CR security

Adversary advantage depends on its

- strategy
- resources: Running time t

Security: H is CR if $\mathbf{Adv}_H^{\mathrm{cr}}(A)$ is "small" for ALL A that use "practical" amounts of resources.

Insecurity: H is insecure (not CR) if there exists A using "few" resources that achieves "high" advantage.

In notes we sometimes refer to CR as CR-KK2.

Example

Let H: $\{0,1\}^k \times \{0,1\}^{256} \rightarrow \{0,1\}^{128}$ be defined by

$$H_K(x) = H_K(x[1]x[2]) = \mathsf{AES}_K(x[1]) \oplus \mathsf{AES}_K(x[2])$$

Is H collision resistant?

Example

Let H: $\{0,1\}^k \times \{0,1\}^{256} \to \{0,1\}^{128}$ be defined by

$$H_K(x) = H_K(x[1]x[2]) = \mathrm{AES}_K(x[1]) \oplus \mathrm{AES}_K(x[2])$$

Is H collision resistant?

Can you design an adversary A

$$K \longrightarrow \boxed{A} \longrightarrow \begin{array}{l} x_1 = x_1[1]x_1[2] \\ x_2 = x_2[1]x_2[2] \end{array}$$

such that $H_K(x_1) = H_K(x_2)$?

Example

Let H: $\{0,1\}^k \times \{0,1\}^{256} \to \{0,1\}^{128}$ be defined by

$$H_K(x) = H_K(x[1]x[2]) = \mathsf{AES}_K(x[1]) \oplus \mathsf{AES}_K(x[2])$$

Weakness:

$$H_K(x[1]x[2]) = H_K(x[2]x[1])$$

adversary A

$x_1 \leftarrow 0^{128}1^{128}$; $x_2 \leftarrow 1^{128}0^{128}$; return x_1, x_2

Then

$$\mathbf{Adv}_H^{\mathrm{cr}}(A) = 1$$

and A is efficient, so H is not CR.

CR-Secure hash functions

- So what might a CR-secure hash function look like?
  - All we know are block ciphers.
  - Let us try the following keyless hash function.

Let $E : \{0,1\}^b \times \{0,1\}^n \rightarrow \{0,1\}^n$ be a block cipher. Let us design keyless compression function

$$h : \{0,1\}^{b+n} \rightarrow \{0,1\}^n$$

by

$$h(x||v) = E_x(v)$$

Is H collision resistant?

NO!

adversary A

Pick some x_1, x_2, v_1 with $x_1 \neq x_2$

$y \leftarrow E_{x_1}(v_1)$; $v_2 \leftarrow E_{x_2}^{-1}(y)$

return $x_1 \| v_1, x_2 \| v_2$

Then

$$E_{x_1}(v_1) = y = E_{x_2}(v_2)$$

CR-Secure hash functions

Let us try this:

Let $E : \{0,1\}^b \times \{0,1\}^n \rightarrow \{0,1\}^n$ be a block cipher. Keyless compression function

$$h : \{0,1\}^{b+n} \rightarrow \{0,1\}^n$$

may be designed as

$$h(x||v) = E_x(v) \oplus v$$

The compression function of SHA1 is underlain in this way by a block cipher $E : \{0,1\}^{512} \times \{0,1\}^{160} \rightarrow \{0,1\}^{160}$.

SHA-1: What does it look like?

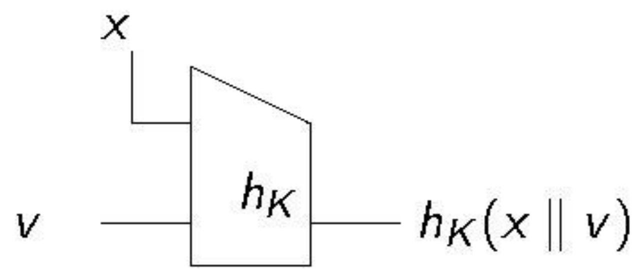
- SHF-1: $\{0,1\}^{128}$ x D -> $\{0,1\}^{160}$
- SHA-1: SHF-1_K,
  where (K = 0x5A827999 || 0x6ED9EBA1 || 0x8F1BBCDC || 0xCA62C1D6)
- SHF-1 and SHA-1 uses a compression function shf1 along with MD transform (Merkle-Damgard).

Compression functions

A compression function is a family $h : \{0,1\}^k \times \{0,1\}^{b+n} \to \{0,1\}^n$ of hash functions whose inputs are of a fixed size $b + n$, where b is called the block size.

E.g. $b = 512$ and $n = 160$, in which case

$$h : \{0,1\}^k \times \{0,1\}^{672} \to \{0,1\}^{160}$$



SHA-1: shf-1

- shf-1_K: $\{0,1\}^{512} \times \{0,1\}^{160} \rightarrow \{0,1\}^{160}$

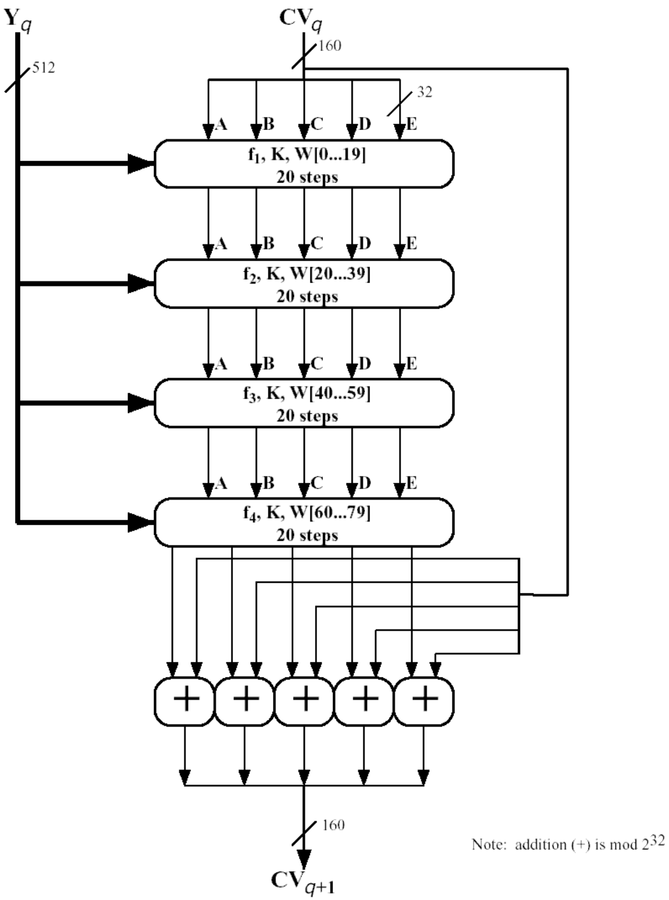


Figure 9.5 SHA-1 Processing of a Single 512-bit Block (SHA-1 Compression Function)

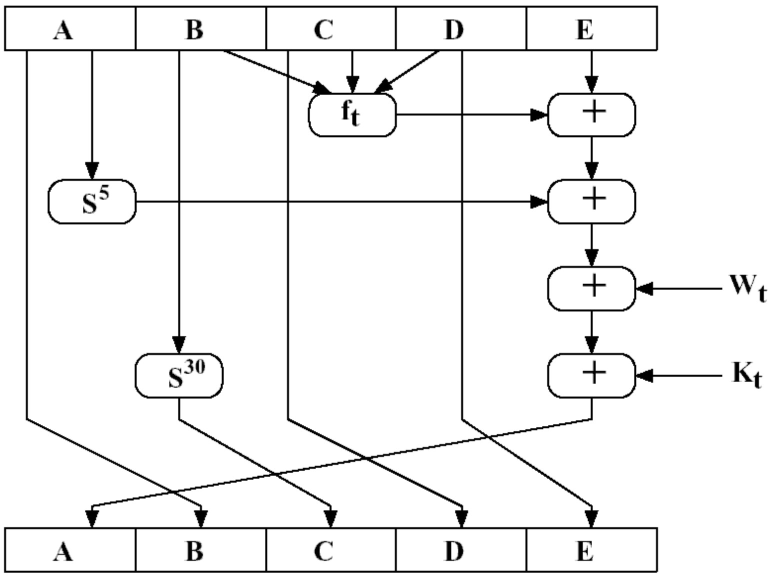


Figure 9.6 Elementary SHA Operation (single step)

Merkle Damgard (MD) Transform

The MD transform

Design principle: To build a CR hash function

$$H : \{0,1\}^k \times D \to \{0,1\}^n$$

where $D = \{0,1\}^{\leq 2^{64}}$:

- First build a CR compression function $h : \{0,1\}^k \times \{0,1\}^{b+n} \to \{0,1\}^n$.
- Appropriately iterate h to get H, using h to hash block-by-block.

MD setup

Assume for simplicity that $|M|$ is a multiple of b. Let

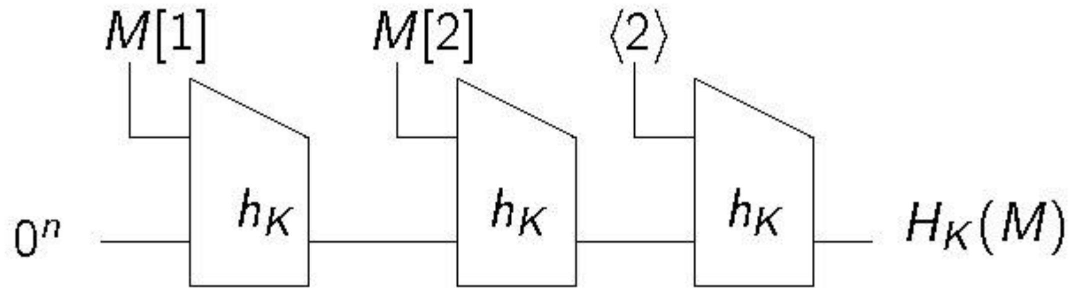
- $\|M\|_b$ be the number of b-bit blocks in M, and write $M = M[1] \ldots M[\ell]$ where $\ell = \|M\|_b$.
- $\langle i \rangle$ denote the b-bit binary representation of $i \in \{0, \ldots, 2^b - 1\}$.
- D be the set of all strings of at most $2^b - 1$ blocks, so that $\|M\|_b \in \{0, \ldots, 2^b - 1\}$ for any $M \in D$, and thus $\|M\|_b$ can be encoded as above.

MD transform

Given: Compression function $h : \{0,1\}^k \times \{0,1\}^{b+n} \to \{0,1\}^n$.

Build: Hash function $H : \{0,1\}^k \times D \to \{0,1\}^n$.

Algorithm $H_K(M)$
$m \leftarrow \|M\|_b$; $M[m+1] \leftarrow \langle m \rangle$; $V[0] \leftarrow 0^n$
For $i = 1, \ldots, m+1$ do $v[i] \leftarrow h_K(M[i] \| V[i-1])$
Return $V[m+1]$



MD preserves CR

Assume

- h is CR
- H is built from h using MD

Then

- H is CR too!

This means

- No need to attack H! You won't find a weakness in it unless h has one
- H is guaranteed to be secure assuming h is.

For this reason, MD is the design used in many current hash functions. Newer hash functions use other iteration methods with analogous properties.

MD preserves CR

Theorem: Let $h : \{0,1\}^k \times \{0,1\}^{b+n} \to \{0,1\}^n$ be a family of functions and let $H : \{0,1\}^k \times D \to \{0,1\}^n$ be obtained from h via the MD transform. Then for any cr-adversary A_H there exists a cr-adversary A_h such that

$$\mathbf{Adv}_H^{\mathrm{cr}}(A_H) \leq \mathbf{Adv}_h^{\mathrm{cr}}(A_h)$$

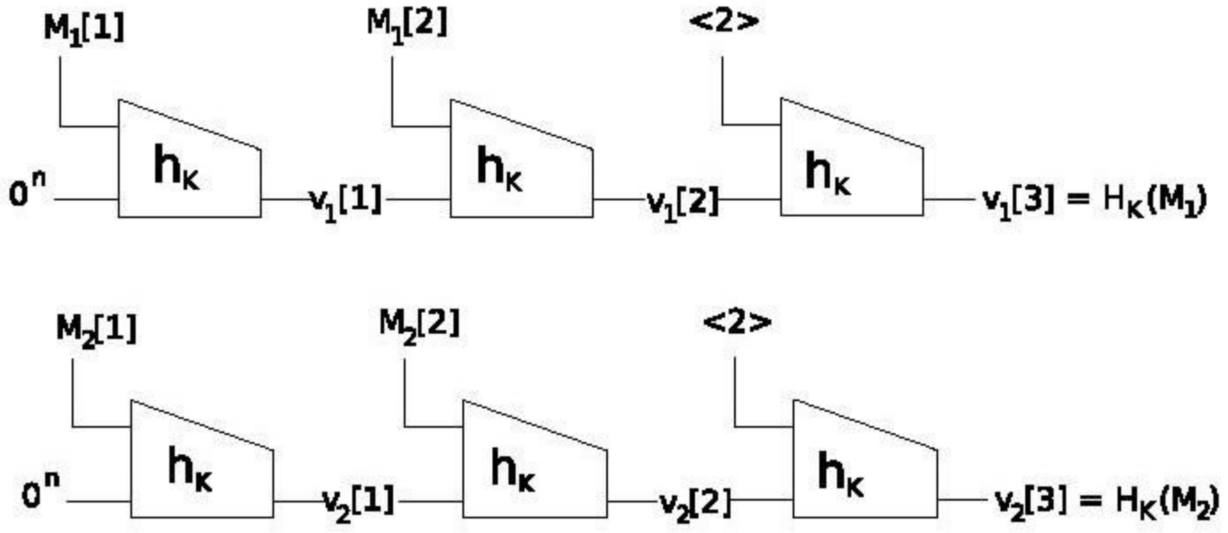
and the running time of A_h is that of A_H plus the time for computing h on the outputs of A_H.

Implication:

$$\begin{aligned} h \ \mathrm{CR} \ &\Rightarrow \ \mathbf{Adv}_H^{\mathrm{cr}}(A_h) \text{ small} \\ &\Rightarrow \ \mathbf{Adv}_H^{\mathrm{cr}}(A_H) \text{ small} \\ &\Rightarrow \ H \ \mathrm{CR} \end{aligned}$$

How does A_h work?

Let (M_1, M_2) be the H_K-collision returned by A_H. The A_h will trace the chains backwards to find an h_K-collision.



Birthday Attack

Attack on the compression function

General collision-finding attacks

We discuss attacks on $H : \{0,1\}^k \times D \to \{0,1\}^n$ that do no more than compute H. Let $D_1, \ldots, D_d$ be some enumeration of the elements of D.

Adversary $A_1(K)$
$x_1 \xleftarrow{\$} D; y \leftarrow H_K(x_1)$
For $i = 1, \ldots, q$ do
    If $(H_K(D_i) = y \wedge x_1 \neq D_i)$ then
        Return x_1, D_i
Return FAIL

Adversary $A_2(K)$
$x_1 \xleftarrow{\$} D; y \leftarrow H_K(x_1)$
For $i = 1, \ldots, q$ do
    $x_2 \xleftarrow{\$} D$
    If $(H_K(x_2) = y \wedge x_1 \neq x_2)$ then
        Return x_1, x_2
Return FAIL

Now:

- A_1 could take $q = d = |D|$ trials to succeed.
- We expect A_2 to succeed in about 2^n trials.

But this still means 2^{160} trials to find a SHA1 collision.

Birthday attacks

Let $H : \{0,1\}^k \times D \to \{0,1\}^n$ be a family of functions with $|D| > 2^n$. The q-trial birthday attack finds a collision with probability about

$$\frac{q^2}{2^{n+1}}.$$

So a collision can be found in about $q = \sqrt{2^{n+1}} \approx 2^{n/2}$ trials.

Recall Birthday Problem

for $i = 1, \ldots, q$ do $y_i \xleftarrow{\$} \{0,1\}^n$
if $\exists i, j$ $(i \neq j$ and $y_i = y_j)$ then $\mathsf{COLL} \leftarrow \mathsf{true}$

$$\begin{aligned} \Pr[\mathsf{COLL}] &= C(2^n, q) \\ &\approx \frac{q^2}{2^{n+1}} \end{aligned}$$

Birthday attack

Let $H : \{0,1\}^k \times D \rightarrow \{0,1\}^n$.

adversary $A(K)$

for $i = 1, \ldots, q$ do $x_i \stackrel{\$}{\leftarrow} D$; $y_i \leftarrow H_K(x_i)$

if $\exists i, j$ $(i \neq j$ and $y_i = y_j$ and $x_i \neq x_j)$ then return x_i, x_j

else return FAIL

Analysis of birthday attack

Let $H : \{0,1\}^k \times D \to \{0,1\}^n$.

adversary $A(K)$
for $i = 1, \ldots, q$ do $x_i \xleftarrow{\$} D$; $y_i \leftarrow H_K(x_i)$
if $\exists i, j$ ($i \neq j$ and $y_i = y_j$ and $x_i \neq x_j$) then return x_i, x_j
else return FAIL

What is the probability that this attack finds a collision?

adversary $A(K)$
for $i = 1, \ldots, q$ do $x_i \xleftarrow{\$} D$; $y_i \leftarrow H_K(x_i)$
if $\exists i, j$ ($i \neq j$ and $y_i = y_j$) then COLL $\leftarrow$ true

We have dropped things that don't much affect the advantage and focused on success probability. So we want to know what is

$$\Pr[\text{COLL}] .$$

Analysis of birthday attack

Birthday

for $i = 1, \ldots, q$ do
  $y_i \xleftarrow{\$} \{0,1\}^n$
if $\exists i, j \ (i \neq j$ and $y_i = y_j)$ then
  $\mathsf{COLL} \leftarrow \mathsf{true}$

$$\Pr[\mathsf{COLL}] = C(2^n, q)$$

Adversary A

for $i = 1, \ldots, q$ do
  $x_i \xleftarrow{\$} D \, ; \, y_i \leftarrow H_K(x_i)$
if $\exists i, j (i \neq j$ and $y_i = y_j)$ then
  $\mathsf{COLL} \leftarrow \mathsf{true}$

$$\Pr[\mathsf{COLL}] = ?$$

Are the two collision probabilities the same?

Not necessarily, because

- on the left $y_i \xleftarrow{\$} \{0,1\}^n$
- on the right $x_i \xleftarrow{\$} D \, ; \, y_i \leftarrow H_K(x_i)$

Analysis of birthday attack

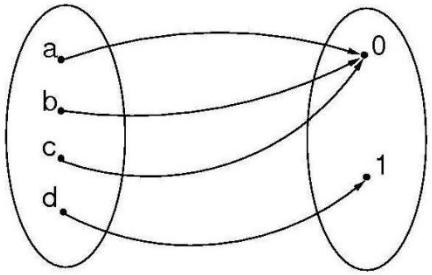
Consider the following processes

Process 1	Process 2
$y \xleftarrow{\$} \{0,1\}^n$	$x \xleftarrow{\$} D;\ y \xleftarrow{\$} H_K(x)$
return y	return y

Process 1 certainly returns a random n-bit string. Does Process 2?

Analysis of birthday attack

Process 1	Process 2
$y \xleftarrow{\$} \{0,1\}$	$x \xleftarrow{\$} \{a,b,c,d\}$; $y \leftarrow H_K(x)$
return y	return y



$$\Pr[y=0] = \frac{1}{2} \qquad \Pr[y=0] = \frac{3}{4}$$

$$\Pr[y=1] = \frac{1}{2} \qquad \Pr[y=1] = \frac{1}{4}$$

Analysis of birthday attack

We say that $H : \{0,1\}^k \times D \rightarrow \{0,1\}^n$ is regular if every range point has the same number of pre-images under H_K. That is if we let

$$H_K^{-1}(y) = \{x \in D : H_K(x) = y\}$$

then H is regular if

$$|H_K^{-1}(y)| = \frac{|D|}{2^n}$$

for all K and y. In this case the following processes both result in a random output

Process 1	Process 2
$y \xleftarrow{\$} \{0,1\}^n$	$x \xleftarrow{\$} D;\ y \xleftarrow{\$} H_K(x)$
return y	return y

Analysis of birthday attack

If H: $\{0,1\}^k \times D \rightarrow \{0,1\}^n$ is regular then the birthday attack finds a collision in about $2^{n/2}$ trials.

Analysis of birthday attack

If H: $\{0,1\}^k \times D \to \{0,1\}^n$ is regular then the birthday attack finds a collision in about $2^{n/2}$ trials.

If H is not regular, the attack may succeed sooner.

So we want functions to be "close to regular".

It seems MD4,MD5,SHA1,RIPEMD,... have this property.

Birthday attack times

Function	n	T_B
MD4	128	2^{64}
MD5	128	2^{64}
SHA1	160	2^{80}
RIPEMD-160	160	2^{80}
SHA256	256	2^{128}

T_B is the number of trials to find collisions via a birthday attack.

Other Attacks

Attacks that may take advantage of the specific construction of the compression function

Cryptanalytic attacks

So far we have looked at attacks that do not attempt to exploit the structure of H.

Can we do better than birthday if we do exploit the structure?

Ideally not, but functions have fallen short!

Cryptanalytic attacks against hash functions

When	Against	Time	Who
1993,1996	md5	2^{16}	[dBBo,Do]
2005	RIPEMD	2^{18}	
2004	SHA0	2^{51}	[JoCaLeJa]
2005	SHA0	2^{40}	[WaFeLaYu]
2005	SHA1	$2^{69}, 2^{63}$	[WaYiYu,WaYaYa]
2009	SHA1	2^{52}	[MHP]
2005,2006	MD5	1 minute	[WaFeLaYu,LeWadW,Kl]

md5 is the compression function of MD5
SHA0 is an earlier, weaker version of SHA1

Status of SHA-1

No collisions yet...

One-Wayness

Another useful notion of security

One-wayness

Let $H : \{0,1\}^k \times D \to \{0,1\}^n$ be a family of functions.

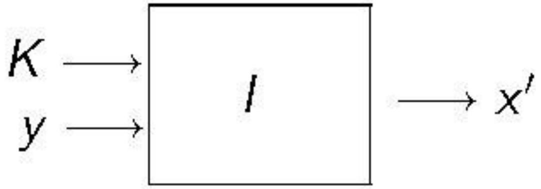
We say that $x' \in D$ is a pre-image of $y \in \{0,1\}^n$ under H_K if $H_K(x') = y$.

Informally: H is one-way if given y and K it is hard to find a pre-image of y under H_K.

One-wayness adversaries

Let $H : \{0,1\}^k \times D \rightarrow \{0,1\}^n$ be a family of functions. A OW - adversary I

- gets input a key K
- gets input some $y = H_K(x) \in D$
- Tries to compute a pre-image of y under H_K



Issues in formalizing one-wayness

Suppose $H_K(0^n) = 0^n$ for all K. Then it is easy to invert H_K at $y = 0^n$ because we know a pre-image of 0^n under H_K: it is simply $x' = 0^n$.

Should this mean H is not one-way?

Turns out what is useful is to ask that it be hard to find a pre-image of the image of a random point.

Formal definition of one-wayness

Let $H : \{0,1\}^k \times D \to \{0,1\}^n$ be a family of functions with D finite, and A a OW-adversary.

Game OW_H

procedure Initialize	procedure Finalize(x')
$K \xleftarrow{\$} \{0,1\}^k$;	return $(H_K(x') = y)$
$x \xleftarrow{\$} D$; $y \leftarrow H_K(x)$	
return K, y	

The ow-advantage of A is

$$\mathbf{Adv}_H^{\mathrm{ow}}(A) = \Pr[\mathrm{OW}_H^A \Rightarrow \mathit{true}].$$

Generic attacks on one-wayness

For any $H : \{0,1\}^k \times D \rightarrow \{0,1\}^n$

- There is an attack that inverts H in about 2^n trials
- But the birthday attack does not apply.

Does CR imply OW?

Let $H : \{0,1\}^k \times D \rightarrow \{0,1\}^n$.

Given: Adversary A attacking one-wayness of H, meaning $A(K, y)$ returns x_2 satisfying $H_K(x_2) = y$.

Want: Adversary B attacking collision resistance of H, meaning $B(K)$ returns x_1, x_2 satisfying $H_K(x_1) = H_K(x_2)$ and $x_1 \neq x_2$.

Adversary $B(K)$

$x_1 \xleftarrow{\$} D;\; y \leftarrow H_K(x_1);\, x_2 \xleftarrow{\$} A(K, y)$

return x_1, x_2

$$\begin{aligned} A \textit{ succeeds} &\Rightarrow H_K(x_2) = y \\ &\Rightarrow H_K(x_2) = H_K(x_1) \\ &\Rightarrow B \textit{ succeeds?} \end{aligned}$$

Problem: May have $x_1 = x_2$.

CR $\not\Rightarrow$ OW

Counter example: Let $H : \{0,1\}^k \times \{0,1\}^n \to \{0,1\}^n$ be defined by

$$H_K(x) = x$$

Then

- H is CR since it is impossible to find $x_1 \neq x_2$ with $H_K(x_1) = H_K(x_2)$.
- But H is not one-way since the adversary A that given K, y returns y has ow-advantage 1.

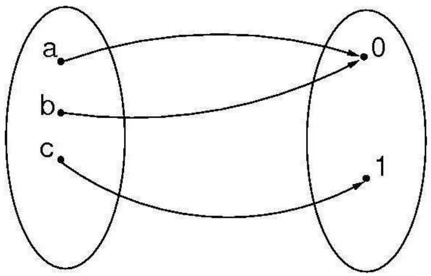
Does CR imply OW?

Adversary $B(K)$

$x_1 \xleftarrow{\$} D;\ \ y \leftarrow H_K(x_1); x_2 \xleftarrow{\$} A(K, y)$

return x_1, x_2

Inuition: If $|D|$ is sufficiently larger than 2^n, meaning H is compressing, then y is likely to have more than one pre-image, and we are likely to have $x_2 \neq x_1$.



In this case, H being CR will imply it is one way

CR ⇒ OW for functions that compress

Theorem: Let $H : \{0,1\}^k \times D \to \{0,1\}^n$ be a family of functions. Let A be a ow-adversary with running time at most t. Then there is a cr-adversary B such that

$$\mathbf{Adv}_H^{\mathrm{ow}}(A) \leq 2 \cdot \mathbf{Adv}_H^{\mathrm{cr}}(B) + \frac{2^n}{|D|}.$$

Furthermore the running time of B is about that of A.

Implication: $\mathrm{CR} \Rightarrow \mathrm{OW}$ as long as $2^n/|D|$ is small.

Proof of Theorem

Adversary $B(K)$
$x_1 \stackrel{\$}{\leftarrow} D;\ \ y \leftarrow H_K(x_1);\ x_2 \stackrel{\$}{\leftarrow} A(K, y)$
return $\ x_1, x_2$

Definition: x_1 is a sibling of x_2 under H_K if x_1, x_2 form a collision for H_K.

For any $K \in \{0,1\}^k$, let

$$S_K = \{x \in D : |H_K^{-1}(H_K(x))| = 1\}$$

be the set of all domain points that have no siblings.

Advantage of B

Adversary $B(K)$

$x_1 \xleftarrow{\$} D$; $y \leftarrow H_K(x_1)$; $x_2 \xleftarrow{\$} A(K, y)$

return x_1, x_2

Then $\mathbf{Adv}_H^{\mathrm{cr}}(B)$

$$
\begin{aligned}
&= \Pr\left[H_K(x_2) = y \wedge x_1 \neq x_2\right] \\
&= \Pr\left[H_K(x_2) = y \wedge x_1 \neq x_2 \wedge x_1 \notin S_K\right] \\
&= \underbrace{\Pr\left[x_1 \neq x_2 \mid H_K(x_2) = y \wedge x_1 \notin S_K\right]}_{1 - \frac{1}{\left|H_K^{-1}(y)\right|} \geq 1 - \frac{1}{2} = \frac{1}{2}} \cdot \Pr\left[H_K(x_2) = y \wedge x_1 \notin S_K\right]
\end{aligned}
$$

Because A has no information about x_1, barring the fact that $H_K(x_1) = y$.

Advantage of B

Adversary $B(K)$

$x_1 \xleftarrow{\$} D$; $y \leftarrow H_K(x_1)$; $x_2 \xleftarrow{\$} A(K, y)$

return x_1, x_2

$$\mathbf{Adv}_H^{\mathrm{cr}}(B) \geq \frac{1}{2}\Pr\left[H_K(x_2) = y \wedge x_1 \notin S_K\right]$$

Fact: $\Pr\left[E \wedge \overline{F}\right] \geq \Pr[E] - \Pr[F]$

Proof: $\Pr\left[E \wedge \overline{F}\right] = \Pr[E] - \Pr[E \wedge F] \geq \Pr[E] - \Pr[F]$

Apply with

$$E : H_K(x_2) = y \quad \text{and} \quad F : x_1 \in S_K$$

$$\mathbf{Adv}_H^{\mathrm{cr}}(B) \geq \frac{1}{2}\left(\Pr[H_K(x_2) = y] - \Pr[x_1 \in S_K]\right)$$

Advantage of B

Adversary $B(K)$

$x_1 \xleftarrow{\$} D;\ \ y \leftarrow H_K(x_1);\ x_2 \xleftarrow{\$} A(K, y)$

return $\quad x_1, x_2$

$$\mathbf{Adv}_H^{\mathrm{cr}}(B) \geq \frac{1}{2}\mathbf{Adv}_H^{\mathrm{ow}}(A) - \frac{\Pr\left[x_1 \in S_K\right]}{2}$$

Recall S_K is the set of domain points that have no siblings, so if $\alpha_1, \alpha_2, \ldots, \alpha_s$ are in S_K then $H_K(\alpha_1), H_K(\alpha_2), \ldots, H_K(\alpha_s)$ must be distinct. So

$$|S_K| \leq |\{0,1\}^n| = 2^n.$$

So

$$\Pr\left[x_1 \in S_K\right] \leq \frac{2^n}{|D|}.$$

Hash Functions

The end