- You have to discuss the running time of your algorithms. Always try to give algorithm with best possible running time.
- You are required to give proofs of correctness whenever needed.
- **Use of unfair means will be severely penalized.**

There are 1 questions for a total of 50 points.

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(50) 1. (*Approximation using LP*) Consider the optimization version of the Vertex Cover problem. That is, given a graph $G = (V, E)$ find a vertex cover of G of minimum size. We know that this problem is NP-hard. Consider the following Integer LP for solving this problem: there are n variables, one for each vertex. Let the variable corresponding to v be denoted by x_v.

LP1
Minimize : $\sum_{v \in V} x_v$,
subject to : $x_u + x_v \geq 1$ for every edge $(u, v) \in E$
$x_v \in \{0, 1\}$ for every vertex $v \in V$

We know that the optimal solution to the above Integer LP would give the minimum sized vertex cover of G. The problem in using this LP to solve the Minimum Vertex Cover problem is that Integer Linear Program is also NP-Hard. Suppose we relax the integer constraint in the above linear program. That is, we consider the following *relaxed* LP:

LP2
Minimize : $\sum_{v \in V} x_v$,
subject to : $x_u + x_v \geq 1$ for each edge $(u, v) \in E$
$x_v \geq 0$ for every vertex $v \in V$

Show the following:

1. Let $(o_1, ..., o_n)$ be an optimal solution of LP2 (these are assignments to the variables). Argue that for all $v \in V, 0 \leq o_v \leq 1$.
2. Suppose we consider the following subset of vertices:
$$S = \{v : o_v \geq 1/2\}$$
Argue that S is a vertex cover of the graph G.
3. Argue that S is at most twice the size of the minimum vertex cover of G. Note that this shows that we have an approximation algorithm for the vertex cover problem that gives approximation factor of 2.
4. Show that the approximation factor of 2 is tight. Recall, we defined tightness when we talked about approximation algorithms using greedy technique.