An Ascending Price Auction for Producer-Consumer Economy

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Abstract

We propose a simultaneous ascending auction mechanism for a producer-consumer economy. A producer-consumer economy comprises of producers, consumers and several types of items. Each producer can produce at most one unit of an item and each consumer wants to consume at most one unit of an item. Each producer's cost of producing each type of item and the valuations of these items by the consumers are private and known to them. For our mechanism we propose simple strategies for producers and consumers which if followed will lead to a nearly efficient allocation. We study the incentive properties of these strategies. Further we show, under certain conditions, if producers and consumers follow these strategies, final prices of our mechanism will converge close to a uniquely defined competitive equilibrium price.

Keywords: Competitive equilibrium; Dominant strategy; English auction; Nash equilibrium; Two sided matching

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1 Introduction

Auctions are well established mechanisms for efficient resource allocation and price discovery. With the popularity of e-commerce on the Internet, there has been a renewed interest in the theory of auctions as well as its applicability in different domains. One such domain is production planning.

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Many a times, producers have machines that can produce a variety of items. In order to effectively use the machine capacities and maximize their profits, the producers need to estimate the demand as well as the selling price for each of the item they can produce. Based on this information, producers can plan their schedule more effectively. However, the actual demand as well as the prices are uncertain and therefore pose a risk to the producers.

To alleviate this problem, capacity auctions are becoming popular. Instead of selling different items, the producers auction capacities to produce different items on their machines. By auctioning these capacities, the producers not only know the exact demand for each of their items, but also the revenues they will obtain by producing these items. This process lets them utilize their resources much more efficiently while minimizing the associated risks.

In this paper we consider a very elementary producer-consumer economy and show how auctions can be used to plan efficient production of items among different competing producers and efficient allocation of these items among different competing consumers. Our contribution is to show how simultaneous ascending price English auctions (Bansal and Garg, 2001; Bansal and Garg, 2002; Demange et al., 1986) can be used effectively in a producer-consumer economy and achieve a nearly efficient allocation. Our most important results are on the strategic behaviors of producers and consumers in such an auction mechanism and their implication on final prices. Unlike other related works (Crawford and Knoer, 1981; Kelso and Crawford, 1982), we derive our results using arbitrary finite bid increments (such as in (Rothkopf and Harstad, 1994)) instead of a suitably small bid increment.

We consider an economy of n consumers, l producers and m items. Each consumer is interested in a subset of m items for which she has positive, private and known valuations. However, each consumer wants at most one item from her subset. Similarly, each producer has a capability to produce a subset of items out of these m items and has a positive and finite production cost associated with each of the items of his subset. However, each of the producer can produce at-most one item at a given time.

Which of the items should a producer produce while avoiding the competition from other producers who can also produce the same item? At what price and to which customer (who has several competing alternatives), and how, should the producer sell his item while maximizing his surplus? The same set of problems are also faced by the consumers. What trading mechanism should be used by the producers and consumers to arrive at an acceptable solution (exchange or trade) that is also fair and efficient? What are the best strategic behaviors of producers and consumers in such a mechanism?

The early work on a related assignment problem was done by Shapley and Shubik (1972) who show that the core of the assignment problem forms a complete lattice. The recent extension of this work by Gul and Stacchetti (1999), who also consider a two-sided market, is more relevant to the producer-consumer problem. They show that if there are "no-complementarities" in production and consumption, then the set of Walrasian prices form a complete lattice. As a result, there is a unique minimum and a unique maximum competitive equilibrium price. Gul and Stacchetti also show that every Walrasian equilibrium assignment is efficient. Single unit demand and single unit supply forms a special case of the "no-complementarity" condition, and hence the results of Gul and Stacchetti are directly applicable to the producer-consumer economy discussed above.

In another paper Gul and Stacchetti (2000) give a double auction mechanism for the producer-consumer economy that converges to the minimum competitive equilibrium price. Gul and Stacchetti assume that the consumer valuations and the producer costs are integer valued and the bid increment is unity. In practice, for a variety of reasons, these assumptions may not be valid.

In this paper we study a more generic auction mechanism, very similar to the simultaneous ascending English auctions (Bansal and Garg, 2002; Demange et al., 1986), that places no restrictions on valuations, costs and bid increments. Such a mechanism is also decentralized in nature and can be implemented across multiple auctioneers in the Internet, with only a minor coordination among them (Bansal and Garg, 2001; Bansal and Garg, 2002). We propose strategies for producers and consumers in this auction mechanism and show that these strategies lead to a nearly efficient allocation. We also give bounds for the inefficiency arising due to discrete and finite bid increments. Earlier Bertsekas (1992) has obtained similar results for a single sided market.

With suitable modifications, the producer-consumer problem can be mapped to a generalized stable marriage problem (Gale and Shapley, 1962; Eriksson and Karlander, 2000) or a two-sided matching market (Demange and Gale, 1985). Demange and Gale study the strategy structure of producers as well as consumers in a two sided matching market in a centralized mechanism. In this mechanism, both sides of the market report their demand and supply structures to a neutral "referee" (or an auctioneer) who calculates the minimum competitive equilibrium and allocates the items accordingly. They show that in such a mechanism, a consumer has no incentive to falsify her demand (or valuations) whereas a producer can obtain better payoffs (upto the payoffs corresponding to the maximum competitive equilibrium price) by falsifying his supply (costs). While these results may be seen as a generalization of second price sealed bid auction mechanism of Vickrey (1961) for a two sided matching market, we obtain similar results for two sided matching markets in the context of ascending English auctions while also taking arbitrary finite bid increments into account. Miyake (1998) has obtained some results on strategic behavior of consumers for a single sided market under some restrictive conditions. Later, this work was generalized by Bansal and Garg (2002) (for single sided markets) while also taking arbitrary finite bid increments into account.

We show how simultaneous independent online auctions (SIA) of Bansal and Garg (2002) can be used effectively in a producer-consumer economy. We extend the LGB strategy proposed by Bansal and Garg to a class of strategies in which LGB(0) refers to the LGB strategy of Bansal and Garg. We also propose a class of greedy strategies for producers called *item selection bidding* (ISB) that uses dummy bidding agents for producers.

We show that when consumers follow the LGB(0) strategy, and producers follow ISB(0) strategy our auction mechanism converges close to the minimum competitive equilibrium. A consumer cannot increase her surplus significantly by following a different strategy, if all other consumers follow the LGB(0) strategy and all the producers follow ISB($\hat{\mathbf{s}}$) strategies with suitable restrictions on $\hat{\mathbf{s}}$. Therefore, the LGB(0) strategy approaches a Nash equilibrium in the producer-consumer economy as the bid increment ϵ approaches zero.

The producers can achieve better surpluses if they set a reserve price higher than their cost of production. However, the producers run into the risk of not being able to sell their capacity as higher reserve prices may not attract any consumer. We show that the maximum achievable surplus of a producer corresponds to his surplus in maximum competitive equilibrium. If a producer sets a reserve price less than or equal to the maximum competitive equilibrium price (minus $(2w+1)\epsilon$, where $w = \min(n, l)$) then he is guaranteed to sell his capacity. On the other hand, a producer will not be able to sell his capacity if he sets a reserve price that exceeds the maximum competitive equilibrium price (by more than $2w\epsilon$), provided all the other producers set a reserve price less than the maximum competitive equilibrium price. So, the strategy of setting a reserve price equal to the maximum competitive equilibrium price (minus $(2w + 1)\epsilon$) approaches a Nash Equilibrium for producers as the bid increment approaches zero. In this case, the final prices in our auction mechanism converge close to the maximum competitive equilibrium price.

In general, if producers set a reserve price higher than their cost of production and lower than the maximum competitive equilibrium price, the final prices in our auction mechanism converge to a uniquely defined competitive equilibrium price. We give a characterization of this price.

The rest of the paper is organized as follows. In Section 2 we describe the ascending price English auction mechanism for the producer-consumer economy with strategies for consumers and producers. In section 3, we discuss the optimality property of our mechanism. Section 4 deals with the strategic behaviors of producers and consumers. In section 5, we investigate the equilibrium properties of the auction mechanism. We conclude in section 6.

2 The Auction Mechanism

Let $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be the set of m types of items that can be produced and consumed in a market comprising a set of producers and consumers. Let $\mathcal{B} = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be the set of n consumers and $\mathcal{G} = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be the set of l producers in the market. Denote the valuation of consumer β_i on item type α_j to be v_{ij} and cost of production of α_j by producer γ_k to be c_{kj}. If consumer β_i is not interested in item α_j then $v_{ij} = 0$. If producer γ_k cannot produce item α_j then c_{kj} is assumed to be infinity. Each producer has capacity to produce a maximum of one item and each consumer wants to get at most one item. We call this the *producer-consumer economy*.

2.1 Auction Rules

We show how simultaneous independent ascending online auctions (Bansal and Garg, 2002) (or multi-item ascending English auctions (Demange et al., 1986)) may be used effectively in the producer-consumer economy. In simultaneous independent ascending auctions (SIA), the consumers bid on items as in ascending English auctions. Since each item can be produced by a subset of $\mathcal{G}$ producers, the consumers bid on $(producer, item)$ tuples. Each of the $(producer, item)$ tuple is auctioned independently as an ascending English auction, except that all the auctions begin and close at the same times. All the producers are required to be present from the beginning till the end of the auctions. The consumers may join or leave the auction at any time, provided that they do not have an outstanding bid on any of the $(producer, item)$ tuple.

Initially, each producer sets a price of zero for each of the item that it can produce. A consumer interested in buying an item from a producer can do so by placing a bid on the corresponding $(producer, item)$ tuple that supersedes the current outstanding bid on the tuple by at-least the minimum bid increment (ϵ). In case there is no outstanding bid on the $(producer, item)$ tuple, a consumer can place a bid that equals the minimum bid increment. The minimum bid increment is usually set by the auctioneer. The auction closes either at a prespecified time, or when there has been no bidding for a reasonable period of time. When the auction closes, the consumer with the highest bid on a $(producer, item)$ tuple pays the amount equal to his bid to the corresponding producer and the producer produces the corresponding item for the consumer.

Denote the value of current outstanding bid at time t on item α_j produced by γ_k as the current price p_{kj} of (γ_k, α_j) at time t. Define the surplus of consumer β_i on tuple (γ_k, α_i) as $v_{ij} - p_{kj}$. Define the surplus of producer γ_k on item α_j as $p_{kj} - c_{kj}$.

2.2 Bidding Strategy of Consumers

We suggest a greedy bidding strategy for consumers as in (Bansal and Garg, 2002). Denote the local greedy bidding strategy with an assured surplus of t as LGB(t). In this strategy, the consumer who does not have an outstanding bid on any ($producer, item$) tuple, examines the prices of all the ($producer, item$) tuples and finds the tuples which give her the maximum surplus. If this surplus is greater than or equal to $t + \epsilon$, then she bids on that particular tuple. The consumer can have more than one tuple which may give her the maximum surplus in which case she arbitrarily chooses one of them. The consumer does not place any more bids in case she has an outstanding (winning) bid on a tuple. LGB refers to a class of strategies with different values of t. If a consumer follows LGB(0) strategy, she will bid on a ($producer, item$) tuple only if her surplus on that tuple is at least ϵ.

LGB strategies make sure that a consumer does not bid on more than one item. Since prices in the auction do not fall, if a consumer has an outstanding bid on any ($producer, item$) tuple, she will not get a better surplus on any other ($producer, item$) tuple in the future. These bidding strategies allow us to state the following two important propositions.

Proposition 1 *If consumer β_i wins item α_j to be produced by γ_k and does not win an item α'_j produced by γ'_k by following an LGB strategy, then $v_{ij} - \hat{p}_{kj} + \epsilon \geq v_{ij'} - \hat{p}_{k'j'}$ where $\hat{\boldsymbol{p}}$ is the final price vector in SIA.*

Proof: Consider the instant when β_i placed the winning bid on a tuple (γ_k, α_j). At that instant, the price of the tuple was $\hat{p}_{kj} - \epsilon$. Let the price of tuple (γ'_k, α'_j) be $p_{k'j'}$ at that instant. Since the consumer follows LGB class of bidding strategies, tuple (γ_k, α_j) gave maximum surplus to β_i at this instant. Therefore, we have $v_{ij} - \hat{p}_{kj} + \epsilon \geq v_{ij'} - p_{k'j'}$. From the rules of the auction, prices of tuple (γ'_k, α'_j) can not drop. Hence we have, $v_{ij} - \hat{p}_{kj} + \epsilon \geq v_{ij'} - \hat{p}_{k'j'}$. ■

Proposition 2 *If a consumer β_i has surplus greater than or equal to $t + \epsilon$ on any ($producer, item$) tuple by following LGB(t) strategy at the end of SIA, then she should get an item.*

Proof: Since β_i has at least $t + \epsilon$ surplus at the end of SIA, on some ($producer, item$) tuple, her maximum surplus at the end of SIA should be at least $t+\epsilon$. From the LGB strategy, she should bid on the ($producer, item$) tuple giving her the maximum surplus, unless she has a highest bid on some other tuple. In that case, since the auction ended here, she did not bid any more. This means she won an item from a producer. ■

2.3 Strategy of Producers

Each producer has a capability to produce a subset of items in $\mathcal{A}$. The consumers may place bids on more than one items from this subset. However, each producer has a capacity to produce only one of the items. Therefore, a producer must ensure that at the end of the auction, there should not be more than one outstanding bid from consumers on the set of items the producer can produce. Each producer ensures this by adding m dummy bidders, one corresponding to each item. As soon as there are more than one outstanding bid by consumers on a producer's items, his dummy bidders outbid all except one of the consumers, according to the producer's strategy.

We propose a greedy bidding strategy, similar to the bidding strategy of consumers, for the dummy bidders of producers. Each producer has a *markup surplus amount* (MSA), $\hat{\mathbf{s}}$. This indicates the surplus the producers want to guarantee for themselves. This can also be thought of as a markup above the cost of production.

Let us call the amount $c_{kj} + \hat{s}_k$, the reserve price of producer γ_k on item α_j. Our suggested bidding strategy ensures that if a producer is assigned he will get his reserve price on the item he is assigned. This is done by the use of dummy agents. If a consumer places a bid on any item of a producer and the bid amount is less than the corresponding reserve price, then the dummy agent outbids that consumer. After a producer receives a bid on an item which is more than his reserve price on that item, the producer ensures that there is just one consumer with an outstanding bid on any of his items. This is ensured by using the dummy agents. The moment consumers place an outstanding bid on more than one item of a producer, his dummy agents outbid the consumers on all the items except on the item giving him the maximum surplus. We call this strategy as *item selection bidding ISB($\hat{\mathbf{s}}$) strategy*. So, ISB refers to a class of strategies for producers and ISB($\hat{\mathbf{s}}$) represents one particular strategy from this class where the MSA for producers is given by the vector $\hat{\mathbf{s}}$. It is easy to see that with ISB strategies, a producer is guaranteed to commit to at most one item at the end of the auction. The following propositions are similar to proposition 1 and proposition 2 and are important properties of the ISB strategy.

Proposition 3 *If a producer γ_k is assigned to produce item α_j in SIA but not α'_j by following an ISB strategy, then $\hat{p}_{kj} - c_{kj} + \epsilon \geq \hat{p}_{kj'} - c_{kj'}$, where $\hat{\boldsymbol{p}}$ is the final price vector in SIA.*

Proof: Consider the instant when the price of tuple (γ_k, α'_j) was $\hat{p}_{kj'} - \epsilon$. Following an ISB strategy, since the dummy agent of the producer places bid on α'_j at this price, either there is some other item α''_j on which he has more surplus than α'_j and on which there is a consumer with the highest bid or the price of none of the items have reached their reserve prices. Since the producer is assigned, in the latter case, at some point in the auction he will have an item which will give him the highest surplus and whose price will be more then the corresponding reserve price. Without loss of generality, let α''_j be that item. Let the price of α''_j when it was the item giving maximum surplus (more than the MSA) to γ_k be $p_{kj''}$. So we have,

$$p_{kj''} - c_{kj''} \geq \hat{p}_{kj'} - c_{kj'} - \epsilon.$$

But later dummy agent of γ_k must have placed bid on α''_j and by ISB strategy, he would have continued his process of selection till he is left with α_j. This means his surplus on α_j must have been more than that on α''_j at that instant. So we can write,

$$\hat{p}_{kj} - c_{kj} \geq p_{kj''} - c_{kj''}.$$

Adding the above two inequalities gives, $\hat{p}_{kj} - c_{kj} + \epsilon \geq \hat{p}_{kj'} - c_{kj'}$. ■

Proposition 4 *If a producer γ_k is unassigned in SIA following ISB($\hat{s}_k$) strategy, then $\hat{p}_{kj} - c_{kj} < \hat{s}_k + \epsilon \ \forall \alpha_j \in \mathcal{A}$, where $\hat{\mathbf{p}}$ is the final price vector in SIA and $\hat{s}_k \geq 0$.*

Proof: If $\hat{p}_{kj} = 0$, the propostion holds. Otherwise, by ISB strategy, dummy agent of producer γ_k would have placed a bid on α_j at price $\hat{p}_{kj} - \epsilon$. Since γ_k is unassigned, by ISB strategy, $\hat{p}_{kj} - \epsilon - c_{kj} < \hat{s}_k$, or $\hat{p}_{kj} - c_{kj} < \hat{s}_k + \epsilon$. ■

2.4 Total Global Surplus

Denote the set of triplets $(\beta_i, \alpha_j, \gamma_k)$ in an assignment by μ. A triplet $(\beta_i, \alpha_j, \gamma_k)$ in μ indicates that item α_j produced by γ_k will be consumed by β_i. Each producer and consumer appears in at most one triplet in an assignment μ. Let us denote the final price vector in the mechanism by $\hat{\mathbf{p}}$. So, item α_j, produced by γ_k, has a final price of $\hat{p}_{kj}$. Let consumer β_i be the winner of α_j to be produced by γ_k. So, the surplus of the consumer on the item α_j is $v_{ij} - \hat{p}_{kj}$. Surplus of the producer on this item is $\hat{p}_{kj} - c_{kj}$. Therefore, surplus of the system due to this assignment is $v_{ij} - c_{kj}$. Define the total global surplus as $S_{SYS} = \sum_{(\beta_i, \alpha_j, \gamma_k) \in \mu} (v_{ij} - c_{kj})$. Thus, the total global surplus of the system is independent of the final price vector. It depends only on the final assignment of items to producers and consumers.

3 Optimality Properties

In this section we prove the optimality of our ascending auction mechanism. In optimal allocation, items will be matched to producers and consumers such that total global surplus of the system is maximized. Let S_{OPT} denote the total global surplus in optimal allocation. Let S_{SYS} denote the total global surplus generated by SIA. Also, let $\hat{p}_{kj}$ denote the final price of item α_j to be produced by γ_k in the SIA mechanism.

Theorem 1 *If producers follow ISB(0) strategy and consumers follow LGB(0) strategy then* $S_{OPT} - S_{SYS} < (l + n)\epsilon$.

The proof is provided in the appendix. The theorem shows that loss in efficiency becomes small as the value of ϵ approaches zero.

4 Strategic Behavior

In the last section we showed that if producers follow ISB(0) strategy and consumers follow LGB(0) strategy, the auction mechanism is efficient as ϵ tends to zero. But what incentives do producers and consumers have to follow these strategies? Can the producer set a higher MSA and still be assured of producing some item? Can a consumer get better surplus by following some other strategy? In this section, we will try to answer these questions. But before we discuss the strategic behavior, we define the notion of competitive equilibrium in this context. The notion of maximum and minimum competitive equilibrium will help us understand the strategic behavior of producers and consumers. Consider a consumer β_i. Define the demand set of β_i at price vector $\mathbf{p}$ as the set of all $(producer, item)$ tuples which give her the maximum non-negative surplus. Mathematically:

$$D_i(\mathbf{p}) = \begin{cases} \phi \text{ if} \max_{(\gamma_k,\alpha_j)\in\mathcal{G}\times\mathcal{A}}[v_{ij} - p_{kj}] < 0, \\ \{(\gamma'_k, \alpha'_j) | v_{ij'} - p_{k'j'} = \max_{(\gamma_k,\alpha_j)\in\mathcal{G}\times\mathcal{A}}[v_{ij} - p_{kj}]\} & \text{otherwise} \end{cases}$$

Similarly we can define the supply set of producer γ_k at price vector $\mathbf{p}$ as the set of all items on which he has the maximum non-negative surplus. Mathematically:

$$S_k(\mathbf{p}) = \begin{cases} \phi \text{ if} \max_{\alpha_j\in\mathcal{A}}[p_{kj} - c_{kj}] < 0, \\ \{\alpha'_j | p_{kj'} - c_{kj'} = \max_{\alpha_j\in\mathcal{A}}[p_{kj} - c_{kj}]\} & \text{otherwise} \end{cases}$$

Now we define the competitive equilibrium prices as follows:

Definition 1 *A price vector* $\mathbf{p}$ *is a competitive equilibrium price if there exists an assignment* μ *containing triplets of the form* $(consumer, item, producer)$ *such that following conditions satisfy:*

- *A producer who is assigned in* μ *is assigned to an item from his supply set and a consumer who is assigned in* μ *is assigned to a* $(producer, item)$ *tuple from her demand set.*

- *All producers with positive surplus on any item are in μ and all producers with negative surplus on all item are not in μ.*
- *All consumers with positive surplus on any $(producer, item)$ tuple are in μ and all consumers with negative surplus on all $(producer, item)$ tuples are not in μ.*
- *Every consumer is assigned at most once. Each producer is assigned at most once.*

The assignment μ is called the competitive equilibrium assignment at $\mathbf{p}$ and the tuple $(\mu, \mathbf{p})$ is called a competitive equilibrium. There can be multiple competitive equilibria. The results of Gul and Stacchetti (1999) imply that the set of competitive equilibrium price vectors form a lattice in the producer-consumer economy. Thus there exists a unique minimum and a unique maximum competitive equilibrium price vector.

Let $\mathbf{p}^{min}$ denote the minimum competitive equilibrium price vector and $\mathbf{p}^{max}$ denote the maximum competitive equilibrium price vector. Let us denote the maximum and minimum surplus vectors for producers as follows. Let $s_{\gamma_k}^{max}$ denote the surplus of producer γ_k on items in his supply set at the maximum competitive equilibrium price $\mathbf{p}^{max}$. It is zero if his supply set is empty. Call $s_{\gamma_k}^{max}$ as the maximum achievable surplus of producer γ_k. Let $\mathbf{s}_p^{max}$ denote the vector of maximum achievable surpluses of producers. Let $s_{\gamma_k}^{min}$ denote the surplus of producer γ_k on items in his supply set (zero if his supply set is empty) at the minimum competitive equilibrium price $\mathbf{p}^{min}$. Call $s_{\gamma_k}^{min}$ as the minimum guaranteed surplus of producer γ_k. Let $\mathbf{s}_p^{min}$ denote the vector of minimum guaranteed surpluses of producers. Let $w = \min(l, n)$.

In order to discuss strategic behaviors in the context of finite bid increments, we need to define the concept of δ-Nash equilibrium.

Definition 2 (δ-Nash equilibrium) *A strategy profile σ^* constitutes a δ-Nash Equilibrium if, for every player i,*

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) - \delta,$$

$\forall\ \sigma_i \in S_i$, *where S_i is the set of strategies player i can adopt and u_i is the utility function of player i.*

4.1 Consumer's Strategy

The following theorem is key in proving that LGB(0) constitutes a $(5w+1)\epsilon$-Nash equilibrium.

Theorem 2 *Let the final price in SIA be $\hat{\mathbf{p}}$ when all the producers follow ISB strategies and all the consumers follow LGB strategies. If β_1 follows* *a different strategy and makes a bid of* $\bar{p}_{11}$ *on a tuple* (γ_1, α_1) *such that* $\bar{p}_{11} < \hat{p}_{11} - 5w\epsilon$ *then* β_1 *will be eventually outbid.*

The proof is provided in Appendix. The SIA mechanism is completely decentralized and (unlike other mechanisms) does not place any restrictions on the order in which consumers or the dummy agents of producers place their bids. Similarly, the LGB and ISB strategies do not place any restrictions on order in which producers and consumers place their bids. However, the order in which bids are placed may alter outcome i.e. the allocations and prices. The following theorem bounds the price variations from one outcome to another.

Theorem 3 *If* $\hat{p}_{kj}$ *and* $\bar{p}_{kj}$ *are the final prices of tuple* (γ_k, α_j) *in two different outcomes of SIA with all the producers following ISB strategies and all the consumers following LGB strategies then* $\forall(k,j) \mid \hat{p}_{kj} - \bar{p}_{kj} \mid\leq 5w\epsilon$.

Proof: If $\hat{p}_{kj} < \bar{p}_{kj}$ then $\bar{p}_{kj} \geq \hat{p}_{kj} - 5w\epsilon$ or else using Theorem 2 the bidder placing a bid of $\bar{p}_{kj}$ in the second outcome will eventually be outbid. Therefore, $\hat{p}_{kj} - \bar{p}_{kj} \leq 5w\epsilon$. Similarly, if $\bar{p}_{kj} < \hat{p}_{kj}$ then $\hat{p}_{kj} - \bar{p}_{kj} \leq 5w\epsilon$. Hence, $\mid \hat{p}_{kj} - \bar{p}_{kj} \mid\leq 5w\epsilon$. ■

Theorem 4 *If all the producers follow ISB strategies and all the consumers follow LGB strategies, then a consumer following the LGB(0) strategy cannot increase her surplus by more than* $(5w+1)\epsilon$ *by following a different strategy.*

Proof: Let $\hat{p}$ be the final prices in one of the outcomes of SIA with all the producers following ISB strategies, consumer β_1 following the LGB(0) strategy and all other consumers following LGB strategies. WLOG, consider a second outcome when β_1 follows a different strategy. Now consider the case when β_1 does not win an item in the second outcome. Since β_1 followed LGB(0) in the first outcome, her surplus is greater than or equal to zero in the first outcome, whereas her surplus is equal to zero in the second outcome. So, in this case β_1 cannot increase her surplus by following a different strategy.

Now consider the case when β_1 wins an item in the second outcome. Let β_1 win the item α_j produced by γ_k at the price $\bar{p}_{kj}$ in the second outcome. Let (γ_1, α_1) be the tuple giving maximum surplus (which may be negative also) to β_1 in the first outcome. Using Theorem 2, $\bar{p}_{kj} \geq \hat{p}_{kj} - 5w\epsilon$. Since, (γ_1, α_1) gives maximum surplus to β_1 in first outcome, $v_{11} - \hat{p}_{11} \geq v_{1j} - \hat{p}_{kj}$. The above two inequalities give, $v_{1j} - \bar{p}_{kj} \leq v_{11} - \hat{p}_{11} + 5w\epsilon$.

If β_1 does not win an item in the first outcome, from proposition 2 and the fact that he follows LGB(0) strategy, we have $v_{11} - \hat{p}_{11} < \epsilon$. This gives us $v_{1j} - \bar{p}_{kj} < (5w+1)\epsilon$. Since the surplus in the first outcome was zero for β_1 in this case, she can not increase her surplus more than $(5w+1)\epsilon$.

If β_1 wins an item, let the tuple that she wins be (γ_2, α_2). By proposition 1, we have $v_{12} - \hat{p}_{22} + \epsilon \geq v_{11} - \hat{p}_{11}$. This gives us $v_{12} - \hat{p}_{12} + (5w+1)\epsilon \geq v_{1j} - \bar{p}_{kj}$. This also implies that she can not increase her surplus by more than $(5w+1)\epsilon$ in the second outcome by following a different strategy. ■

The above theorem implies that LGB(0) constitutes a $(5w+1)\epsilon$-Nash equilibrium for consumers in the producer-consumer economy with SIA mechanism and when the producers follow ISB strategies. The above theorem also implies that if the set of consumer strategies are restricted to LGB class of strategies, then LGB(0) approaches a dominant strategy as the bid increment approaches zero.

4.2 Producer's Strategy

The producer's strategic behavior is remarkably different from that of consumers in the producer-consumer economy with SIA mechanism. For consumers, LGB(0), the honest strategy of bidding according to the true valuations approaches Nash equilibrium as the minimum bid increment approaches zero. However, producers can realize better surpluses by setting their reserve prices above their true cost of production. The producers run into the risk of loosing consumers it they raise the reserve prices too much. The following two theorems illustrate the maximum reserve price that may be set by the producers without loosing their consumers.

Theorem 5 *Let producers follow ISB($\hat{s}$) strategy ($\hat{s} \geq 0$), and consumers follow LGB(0) strategy . If $0 \leq \hat{s}_1 \leq s^{max}_{\gamma_1} - (2w+1)\epsilon$, then γ_1 will produce some item in SIA irrespective of the markup surplus amounts of other producers.*

The proof is provided in the Appendix. This theorem throws some light into the optimality property of SIA mechanism also. If all producers follow ISB($\hat{\mathbf{s}}$) strategy with $0 \leq \hat{\mathbf{s}} \leq \mathbf{s}^{max} - (2w+1)\epsilon$, then they will be assigned some item. Our proof of Theorem 1 will still hold good in that case. The only place we use the fact that producers should follow ISB(0) strategy is while deriving Equation 1 in the proof. If all producers are assigned and consumers follow LGB(0) strategy, then this derivation does not require producers to follow ISB(0) strategy. So SIA leads to nearly efficient allocation when consumers follow LGB(0) strategy and producers follow ISB($\hat{\mathbf{s}}$) strategy with $0 \leq \hat{\mathbf{s}} \leq \mathbf{s}^{max} - (2w+1)\epsilon$.

Theorem 6 *Let all producers follow ISB($\hat{s}$) strategy and all consumers follow LGB(t) strategy ($t \geq 0$). If $\hat{s}_1 \geq s^{max}_{\gamma_1} + 2w\epsilon$ and $\hat{s}_{-1} \leq s^{max}_{-\gamma_1}$, then γ_1 will not produce any item in SIA.*

The proof is provided in the Appendix.

The above two theorems also indicate that if the set of producer strategies are restricted to ISB, then the strategy ISB($\mathbf{s}_p^{max} - (2w+1)\epsilon$) constitutes a $(4w+1)\epsilon$-Nash equilibrium for the producers.

In general, the producers will not know their maximum achievable surplus ($\mathbf{s}_p^{max}$). In order to compute this, they need to know the cost of production of all other producers and the valuations of all the consumers. In view of this Theorem 5 holds a special significance. Theorem 5 indicates that if a producer makes a conservative estimate of his maximum achievable surplus, he is guaranteed to produce an item and utilize his capacity. The following section characterizes the final prices in SIA if all the producers make conservative estimates of their maximum achievable surplus.

5 Competitive Equilibrium

We denote the surplus of producer γ_k on the items in his supply set at price vector $\mathbf{p}$ as $s_{\gamma_k}(\mathbf{p})$ (zero if the supply set is the null set). We denote the surplus vector of all the producers as $\mathbf{s}(\mathbf{p})$. As before $\hat{s}_{\gamma_k}$ denotes the markup surplus amount of producer γ_k and the vector of all markup surplus amounts of all producers is denoted by $\hat{\mathbf{s}}$. Define a "minimal competitive equilibrium price at $\mathbf{s}$" as follows:

Definition 3 *$\boldsymbol{p}$ is a minimal competitive equilibrium price vector at $\boldsymbol{s}$ if it is a competitive equilibrium price vector such that $\boldsymbol{s}(\boldsymbol{p}) \geq \boldsymbol{s}$ and there does not exist another competitive equilibrium price vector $\tilde{\boldsymbol{p}} < \boldsymbol{p}$ with $\boldsymbol{s}(\tilde{\boldsymbol{p}}) \geq \boldsymbol{s}$.*

So minimal competitive equilibrium price vector is the smallest competitive equilibrium price vector whose corresponding surplus vector is greater than or equal to the given surplus vector. For example if $\mathbf{s}$ is a vector of zeros, then the minimal competitive equilibrium price at $\mathbf{s}$ is unique and corresponds to the minimum competitive equilibrium price vector (Gul and Stacchetti, 1999). We generalize this result to arbitrary surplus vectors less than or equal to maximum achievable surplus vector of producers, $\mathbf{s}^{max}$, which corresponds to the surplus of producers on items in their supply set in the maximum competitive equilibrium.

Theorem 7 *For any surplus vector $\boldsymbol{s} \leq \boldsymbol{s}_p^{max}$, the minimal competitive equilibrium price vector at $\boldsymbol{s}$ (denoted by $\boldsymbol{p}^{min}(\boldsymbol{s})$) exists and is unique.*

The proof is provided in the Appendix.

Since the minimal competitive equilibrium price at $\mathbf{s}$ is unique, we call it the minimum competitive equilibrium price vector at $\mathbf{s}$ and denote it as $\mathbf{p}^{min}(\mathbf{s})$. The next two theorems show that if the producers follow ISB($\hat{\mathbf{s}}$) strategy with $0 \leq \hat{\mathbf{s}} \leq \mathbf{s}_p^{max}$ and consumers follow LGB(0) strategy, then the final prices will converge close to the minimum competitive equilibrium price at $\mathbf{s}$, $\mathbf{p}^{min}(\hat{\mathbf{s}})$.

Theorem 8 *If $\hat{\mathbf{p}}$ is the final price vector in SIA, when all the producers follow ISB($\hat{\mathbf{s}}$) strategies with $0 \leq \hat{\mathbf{s}} \leq \mathbf{s}_p^{max}$ and all consumers follow LGB(0) strategy, then $\forall(\gamma_k, \alpha_j)\hat{p}_{kj} \geq p_{kj}^{min}(\hat{\mathbf{s}}) - 2(w+1)\epsilon$.*

The proof is provided in the Appendix.

Theorem 9 *If $\hat{\mathbf{p}}$ is the final price vector in SIA, when all the producers follow ISB($\hat{\mathbf{s}}$) strategies with $0 \leq \hat{\mathbf{s}} \leq \mathbf{s}_p^{max}$ and all consumers follow LGB(0) strategy, then $\forall\ (\gamma_k, \alpha_j), \hat{p}_{kj} - p_{kj}^{min}(\hat{\mathbf{s}}) \leq (4w+3)\epsilon$.*

So, in the extreme cases, if the reserve prices are set to the true cost of production in SIA (with consumers following the LGB(0) strategy), then the final prices will converge close to the minimum competitive equilibrium prices. On the other extreme, if each producer γ having $s_\gamma^{max} \geq (2w+1)\epsilon$, sets a markup surplus amount $\hat{s}_\gamma = s_\gamma^{max} - (2w+1)\epsilon$, then in SIA he will be assigned to produce an item, and the final prices will converge close to the maximum competitive equilibrium prices.

Gul and Stacchetti (1999) showed that in any competitive equilibrium, prices of all $(producer, item)$ tuples involving the same item are equal. Below we provide a theorem which states a result analogous to theirs.

Theorem 10 *For any two producers, γ_1 and γ_2, let $\hat{p}_{11}$ and $\hat{p}_{21}$ be the prices of tuples (γ_1, α_1) and (γ_2, α_1) respectively in SIA when consumers follow LGB(0) strategy and producers follow ISB strategy. $|\hat{p}_{11} - \hat{p}_{21}| \leq (6w+5)\epsilon$.*

Proof: Let $\mathbf{p}$ denote the minimal competitive equilibrium price at $\hat{\mathbf{s}}$, where $\hat{\mathbf{s}}$ is the vector of MSAs of producers. From Theorem 9, $\hat{p}_{11} \leq p_{11} + (4w+3)\epsilon$. From Gul and Stacchetti (1999), we get $p_{11} = p_{21}$. So, $\hat{p}_{11} \leq p_{21} + (4w+3)\epsilon$. But from Theorem 8, $p_{21} \leq \hat{p}_{21} + (2w+2)\epsilon$. This gives us $\hat{p}_{11} \leq \hat{p}_{21} + (6w+5)\epsilon$. Similarly we can show that $\hat{p}_{21} \leq \hat{p}_{11} + (6w+5)\epsilon$. ■

6 Conclusion

In this paper we considered a simple producer-consumer economy where there is a set of producers, a set of consumers and a set of items. In our model each producer is capable of producing a subset of items, but has a capacity to produce only one item at a time. Similarly the consumers are interested in a subset of items, but want to get at most one item from their subset. We assume that both producers and consumers have private and known valuations.

We show how the simultaneous independent ascending auction mechanism (SIA) (Bansal and Garg, 2002) can be used in this economy. We proposed strategies for producers and consumers for this mechanism. The suggested strategies result in efficient production-consumption assignment. As a result the total global surplus in the economy is maximized.

Our main contribution is the study of strategic behaviors of producers and consumers in the producer-consumer economy (that can easily be mapped to a two-sided matching market) with simultaneous independent ascending auction mechanism with arbitrary finite minimum bid increment. We show that the truth-telling greedy bidding strategy is close to a Nash equilibrium for consumers. We also show that setting reserve prices close to the maximum competitive equilibrium prices in the producer-consumer economy approaches a Nash equilibrium for producers as the minimum bid increment approaches zero.

Since the producers do not know the maximum competitive equilibrium price in advance, they need to make an estimate which exposes them to a risk. We argue that a risk free strategy for producers is to make a conservative estimate of the maximum competitive equilibrium price. We characterize the final prices in the mechanism (which are always Walrasian) if all the producers make a conservative estimate of the maximum competitive equilibrium price.

Our contribution holds a significant value because of the popularity of auctions and e-commerce over the Internet. The decentralized nature of simultaneous independent auctions, makes it possible to implement them across multiple Internet auction sites without requiring a significant coordination among them. This could create integrated auction markets spanning the entire Internet. Thus, producers with limited machine capacities, capable of producing multiple items, can utilize their capacities more effectively using simultaneous auctions in the Internet. Our results on strategic behaviors enables the producers as well as the consumers to choose effective bidding strategies that maximize their surpluses. The use of automated bidding agents (also referred to as proxy bidding) will lead to an automated implementation of our proposed strategies. These agents also let the auctioneers to choose a very small bid increment thus minimizing the inefficiency in the process.

References

Bansal, V. and Garg, R., 2001, Efficiency and Price Discovery in Multi-item Auctions, ACM SIGecom Exchanges 2(1)

Bansal, V. and Garg, R., 2002, Simultaneous Independent Online Auctions with Discrete Bid Increments, Electronic Commerce Research Journal: Special issue on Dynamic Pricing Policies in Electronic Commerce

Bertsekas, D. P., 1992, Auction Algorithms for Network Flow Problems: A Tutorial Introduction, Computational Optimization and Applications 1, 7–66

Crawford, V. P. and Knoer, E. M., 1981, Job Matching with Heterogeneous Firms and Workers, Econometrica 49(2), 437–450

Demange, G. and Gale, D., 1985, The Strategy Structure of Two-sided Matching Markets, Econometrica 53, 873–888

Demange, G., Gale, D., and Sotomayor, M., 1986, Multi-item Auctions, Journal of Political Economy 94(4), 863–872

Eriksson, K. and Karlander, J., 2000, Stable matching in a common generalization of the marriage and assignment models, Discrete Math. 217(1-3), 135–156

Gale, D. and Shapley, L., 1962, College Admissions and the Stability of Marriage, American Mathematical Monthly 69, 9–15

Gul, F. and Stacchetti, E., 1999, Walrasian Equilibrium with Gross Substitutes, Journal of Economic Theory 87(1), 95–124

Gul, F. and Stacchetti, E., 2000, The English Auction with Differentiated Commodities, Journal of Economic Theory 92(1), 66–95

Kelso, A. S. and Crawford, V. P., 1982, Job Matching, Coalition Formation, and Gross Substitutes, Econometrica 50(6), 1483–1504

Miyake, M., 1998, On the Incentive Properties of Multi-item Auctions, International Journal of Game Theory 27(1), 1–19

Rothkopf, M. H. and Harstad, R. M., 1994, On the Role of Discrete Bid Levels in Oral Auctions, European Journal of Operational Research 74, 572–581

Shapley, L. S. and Shubik, M., 1972, The Assignment Game I: The Core, International Journal of Game Theory 1(2), 111–130

Vickrey, W., 1961, Counterspeculation, Auctions, and Competitive Sealed Tenders, Journal of Finance 16, 8–37

Appendix

Before proving our main theorems, we introduce the notion of a *preferred bid*. In SIA mechanism for the producer-consumer economy, consumers place bids on $(producer, item)$ tuple. Producers following ISB strategy use dummy bidding agents to make sure that only one item, that gives them the maximum surplus, has an outstanding bid from a consumer, after the price of any of their items has reached its reserve price. A new bid placed by a consumer is called a *preferred bid* if it increases the surplus of the corresponding producer and gives the producer more surplus than his MSA.

Proof **(Theorem 1):** Let B_{OPT} and G_{OPT} be the sets of consumers and producers respectively that were assigned in optimal matching. So we can write the total global surplus in optimal matching as

$$S_{OPT} = \sum_{\beta_i \in B_{OPT}} (v_{ij} - c_{kj}).$$

where β_i is assigned item α_j to be produced by producer γ_k. Denoting the final price in SIA by $\hat{\mathbf{p}}$, the above equation can be rewritten as

$$S_{OPT} = \sum_{\beta_i \in B_{OPT}} (v_{ij} - \hat{p}_{kj} + \hat{p}_{kj} - c_{kj}).$$

or,

$$S_{OPT} = \sum_{\beta_i \in B_{OPT}} (v_{ij} - \hat{p}_{kj}) + \sum_{\gamma_k \in G_{OPT}} (\hat{p}_{kj} - c_{kj}).$$

Now let us divide the set B_{OPT} to two subsets. B_{BOTH} is the set of consumers who are assigned in optimal matching as well as in SIA and $\hat{B}$ is the set of consumers who are assigned only in optimal matching and not in SIA. Similarly we can define G_{BOTH} and $\hat{G}$. With this the above equation can be rewritten as

$$S_{OPT} = \sum_{\beta_i \in B_{BOTH}} (v_{ij} - \hat{p}_{kj}) + \sum_{\beta_i \in \hat{B}} (v_{ij} - \hat{p}_{kj})$$

$$+ \sum_{\gamma_k \in G_{BOTH}} (\hat{p}_{kj} - c_{kj}) + \sum_{\gamma_k \in \hat{G}} (\hat{p}_{kj} - c_{kj}).$$

By proposition 2 and LGB(0) strategy, if a consumer β_i is not assigned in SIA, $v_{ij} - \hat{p}_{kj} < \epsilon \ \forall (\gamma_k, \alpha_j)$. Similarly by ISB(0) strategy and proposition 4, if a producer γ_k is not assigned an item in SIA, $\hat{p}_{kj} - c_{kj} < \epsilon \ \forall \alpha_j \in \mathcal{A}$. Denoting, $|\hat{B}| = q_1$ and $|\hat{G}| = t_1$, we transform the previous equation as

$$S_{OPT} < \sum_{\beta_i \in B_{BOTH}} (v_{ij} - \hat{p}_{kj}) + q_1 \epsilon + \sum_{\gamma_k \in G_{BOTH}} (\hat{p}_{kj} - c_{kj}) + t_1 \epsilon. \quad (1)$$

We can define the total global surplus in SIA mechanism, S_{SYS}, in the similar lines using sets B_{SYS} and G_{SYS}.

$$S_{SYS} = \sum_{\beta_i \in B_{SYS}} (v_{ij'} - \hat{p}_{kj'}) + \sum_{\gamma_k \in G_{SYS}} (\hat{p}_{kj'} - c_{kj'}).$$

where β_i is assigned item α'_j to be produced by γ_k in SIA. As before B_{SYS} can be divided into two sets, B_{BOTH} and $\bar{B}$, where B_{BOTH} is the set of consumers assigned in both SIA and optimal matching and $\bar{B}$ is the set of consumers assigned in SIA and not in optimal matching. Similarly we can define $\bar{G}$ and G_{BOTH}(already defined earlier). Now the previous equation can be written as

$$S_{SYS} = \sum_{\beta_i \in B_{BOTH}} (v_{ij'} - \hat{p}_{kj'}) + \sum_{\beta_i \in \bar{B}} (v_{ij'} - \hat{p}_{kj'})$$

$$+ \sum_{\gamma_k \in G_{BOTH}} (\hat{p}_{kj'} - c_{kj'}) + \sum_{\gamma_k \in \bar{G}} (\hat{p}_{kj'} - c_{kj'}).$$

By ISB and LGB strategies, all the producers and consumers who are assigned have positive surplus. So if $\beta_i \in \bar{B}$, we can write $v_{ij'} - \hat{p}_{kj'} \geq 0$, where β_i is assigned item α'_j to be produced by γ_k in SIA. Similarly, we can write, $\hat{p}_{kj'} - c_{kj'} \geq 0$, for $\gamma_k \in \bar{G}$. So the last equation can be rewritten as

$$S_{SYS} \geq \sum_{\beta_i \in B_{BOTH}} (v_{ij'} - \hat{p}_{kj'}) + \ 0 \ + \sum_{\gamma_k \in G_{BOTH}} (\hat{p}_{kj'} - c_{kj'}) + \ 0$$

or

$$S_{SYS} = \sum_{\beta_i \in B_{BOTH}} (v_{ij'} - \hat{p}_{kj'}) + \sum_{\gamma_k \in G_{BOTH}} (\hat{p}_{kj'} - c_{kj'}). \tag{2}$$

By equations 1 and 2, we can write $S_{OPT} - S_{SYS}$ as follows

$$S_{OPT} - S_{SYS} < \sum_{\beta_i \in B_{BOTH}} (v_{ij} - \hat{p}_{kj} - v_{ij'} + \hat{p}_{k'j'})$$

$$+ \sum_{\gamma_k \in G_{BOTH}} (\hat{p}_{kj} - c_{kj} - \hat{p}_{kj'} + c_{kj'}) + q_1\epsilon + t_1\epsilon.$$

Let $|B_{BOTH}| = q_2$ and $|G_{BOTH}| = t_2$. Applying proposition 1 to every term in the first summation and proposition 3 to every term in the second summation, we have

$$S_{OPT} - S_{SYS} < q_2\epsilon + t_2\epsilon + q_1\epsilon + t_1\epsilon$$

The maximum value of $q_1 + q_2$ is n and the maximum value of $t_1 + t_2$ is l. So we can write $S_{OPT} - S_{SYS} < (l + n)\epsilon$. ■

Proof **(Theorem 2):** Let us denote the outcome when every consumer follows an LGB(t) strategy as X. Let us denote the outcome when β_1 deviates, while others follow an LGB(t) strategy as Y. Let us denote the final price of SIA in Y as $\bar{\mathbf{p}}$. Assume for contradiction, β_1 is not outbid in Y, by placing her bid. This means (γ_1, α_1) will be assigned in Y to β_1 at price $\bar{p}_{11}$. Since $\bar{p}_{11} < \hat{p}_{11} - 5w\epsilon$, γ_1 will have more surplus in X than in Y. Since he is assigned in Y by following ISB, he will also be assigned in X. Consider the following lemma now.

Lemma 1 *Let $\mathcal{I}$ be a set of consumers assigned in Y to a set of producers, denoted by $\mathcal{J}$, such that if $\beta_i \in \mathcal{I}$ is assigned to (γ_2, α_2), then $\bar{p}_{22} < \hat{p}_{22} - \delta$ ($\delta \geq 5\epsilon$). There exists a producer $\gamma_3 \notin \mathcal{J}$ which is assigned an item α_3 to be consumed by $\beta_2 \notin \mathcal{I}$ in Y such that $\bar{p}_{33} < \hat{p}_{33} - \delta + 5\epsilon$.*

Proof: Without loss of generality(WLOG), consider a producer $\gamma_1 \in \mathcal{J}$ who is assigned an item α_1 in Y. Since prices in X, of items produced by the producers in $\mathcal{J}$, are more than their prices in Y by δ, and $\delta \geq 5\epsilon$, and the producers follow ISB strategies, γ_1 will be assigned in X. So all the producers in $\mathcal{J}$ will be assigned in X. Consider the following two cases.

Case 1: There exists a producer $\gamma_2 \in \mathcal{J}$ who is assigned to (β_2, α_2) in X, and $\beta_2 \notin \mathcal{I}$. Let the assignment of γ_2 in Y involve α_1. In both X and Y, all producers follow ISB strategy. So, applying proposition 3 to γ_2 in X, we have,

$$\hat{p}_{22} - c_{22} + \epsilon \geq \hat{p}_{21} - c_{21}.$$

Applying proposition 3 to γ_2 in Y, we have,

$$\bar{p}_{21} - c_{21} + \epsilon \geq \bar{p}_{22} - c_{22}.$$

Adding the above two equations and using the fact that $\hat{p}_{21} - \bar{p}_{21} > \delta$, we get $\hat{p}_{22} - \bar{p}_{22} + 2\epsilon \geq \hat{p}_{21} - \bar{p}_{21} > \delta$. So we get $\hat{p}_{22} - \bar{p}_{22} > \delta - 2\epsilon$. Since $\beta_2 \notin \mathcal{I}$, this implies $\beta_2 \neq \beta_1$. This means β_2 follows some LGB(t) strategy. So $v_{22} - \hat{p}_{22} \geq t$. This gives us $v_{22} - \bar{p}_{22} > t + \delta - 2\epsilon$. Since $\delta \geq 5\epsilon$, by proposition 2, β_2 will be assigned in Y. Let her assignment in Y be (γ_3, α_3). Clearly, $\gamma_3 \notin \mathcal{J}$. Applying proposition 1 to β_2 in X, we have,

$$v_{22} - \hat{p}_{22} + \epsilon \geq v_{23} - \hat{p}_{33}.$$

Applying proposition 1 to β_2 in Y, we have,

$$v_{23} - \bar{p}_{33} + \epsilon \geq v_{22} - \bar{p}_{22}$$

Adding the above two equations and using the fact that $\hat{p}_{22} - \bar{p}_{22} > \delta - 2\epsilon$, we get, $\hat{p}_{33} - \bar{p}_{33} + 2\epsilon \geq \hat{p}_{22} - \bar{p}_{22} > \delta - 2\epsilon$. So we have, $\hat{p}_{33} - \bar{p}_{33} > \delta - 4\epsilon$.

Case 2: All producers in $\mathcal{J}$ are assigned to consumers in $\mathcal{I}$ in X. Let β_2 be the consumer who placed a *preferred bid* on one of the items produced by producers in $\mathcal{J}$, and her bid was the last such bid that was outbid. Easy to see that $\beta_2 \notin \mathcal{I}$. Let $\gamma_2 \in \mathcal{J}$ be the corresponding producer. Observe that after β_2 has been outbid, no other *preferred bid* will be placed on items produced by producers of $\mathcal{J}$. So β_2 is outbid by either a consumer or the producer agent of γ_2. We consider both the cases.

Case 2(a): β_2 is outbid by another consumer β_3. This means β_3 is assigned γ_2 in X. Let the item in this assignment be α_2. Thus, β_2 bids on (γ_2, α_2) when its price was $\hat{p}_{22} - 2\epsilon$. Since $\beta_2 \notin \mathcal{J}$, $\beta_2 \neq \beta_1$. This implies that β_2 follows an LGB(t) strategy in X as well as in Y. Let the assignment of γ_2 in Y involve item α_1. Applying proposition 3 on γ_2 in X, we have,

$$\hat{p}_{22} - c_{22} + \epsilon \geq \hat{p}_{21} - c_{21}.$$

Applying proposition 3 on γ_2 in Y, we have,

$$\bar{p}_{21} - c_{21} + \epsilon \geq \bar{p}_{22} - c_{22}.$$

Adding the above two equations and using the fact that $\hat{p}_{21} - \bar{p}_{21} > \delta$, we get $\hat{p}_{22} - \bar{p}_{22} + 2\epsilon \geq \hat{p}_{21} - \bar{p}_{21} > \delta$. This gives us $\hat{p}_{22} - \bar{p}_{22} > \delta - 2\epsilon$. Since β_2 follows an LGB(t) strategy while bidding on (γ_2, α_2) in X, we have $v_{22} - \hat{p}_{22} + 2\epsilon \geq t$. Therefore, $v_{22} - \bar{p}_{22} > t + \delta - 4\epsilon$. Since $\delta \geq 5\epsilon$, β_2 will be assigned in X by proposition 2. Let her assignment in Y involve (γ_3, α_3). Clearly $\gamma_3 \notin \mathcal{J}$. Let the price of (γ_3, α_3) be p_{33} when β_2 placed her last bid on (γ_2, α_2) in X. Since β_2 follows an LGB strategy and prices in SIA can not decrease,

$$v_{22} - \hat{p}_{22} + 2\epsilon \geq v_{23} - p_{33} \geq v_{23} - \hat{p}_{33}.$$

Applying proposition 1 on β_2 in Y,

$$v_{23} - \bar{p}_{33} + \epsilon \geq v_{22} - \bar{p}_{22}.$$

Adding the above two equations we get $\hat{p}_{33} - \bar{p}_{33} + 3\epsilon \geq \hat{p}_{22} - \bar{p}_{22}$. But we have shown $\hat{p}_{22} - \bar{p}_{22} > \delta - 2\epsilon$. So we get $\hat{p}_{33} - \bar{p}_{33} > \delta - 5\epsilon$.

Case 2(b): β_2 gets outbid by the producer agent of γ_2. Let the item on which β_2 gets outbid be α_4. Let the assignment of γ_2 in X involve α_2. Let the assignment of γ_2 in Y involve item α_1. Consider the instant in X when β_2 had the last outstanding bid on (γ_2, α_4). The price of (γ_2, α_4) at this instant was $\hat{p}_{24} - \epsilon$. Since β_2 placed a *preferred bid* on an item produced by γ_2 and was the last such consumer that had been outbid, later in X, the surplus of γ_2 can only increase by atmost ϵ. This implies, $(\hat{p}_{24} - \epsilon) - c_{24} + \epsilon \geq s_{\gamma_2}$. Let s_{γ_2} be the surplus of γ_2 at the end of auction in X. Applying proposition 3, we get $s_{\gamma_2} \geq \hat{p}_{21} - c_{21} - \epsilon$. Therefore,

$$(\hat{p}_{24} - \epsilon) - c_{24} + \epsilon \geq s_{\gamma_2} \geq \hat{p}_{21} - c_{21} - \epsilon$$

Applying proposition 3 to γ_2 in Y, we get,

$$\bar{p}_{21} - c_{21} + \epsilon \geq \bar{p}_{24} - c_{24}.$$

From the above two equations we get $\hat{p}_{24} - \bar{p}_{24} + 2\epsilon \geq \hat{p}_{21} - \bar{p}_{21}$. But we know $\hat{p}_{21} - \bar{p}_{21} > \delta$. So we get $\hat{p}_{24} - \bar{p}_{24} > \delta - 2\epsilon$. Since β_2 bids on (γ_2, α_4) in X by following LGB(t) strategy at price $\hat{p}_{24} - 2\epsilon$, $v_{24} - \hat{p}_{24} + 2\epsilon \geq t$. This gives us $v_{24} - \bar{p}_{24} > t + \delta - 4\epsilon$. Since $\delta \geq 5\epsilon$, from proposition 2, β_2 will be assigned in Y. Let her assignment in Y involve (γ_3, α_3). In this case since all the producers in $\mathcal{J}$ are assigned to consumers in $\mathcal{I}$, and $\beta_2 \notin \mathcal{I}$, therefore $\gamma_3 \notin \mathcal{J}$. Let the price of (γ_3, α_3), when β_2 places her last bid on (γ_2, α_4) in X, be p_{33}. By LGB strategy,

$$v_{24} - \hat{p}_{24} + 2\epsilon \geq v_{23} - p_{33} \geq v_{23} - \hat{p}_{33}.$$

Applying proposition 1 to β_2 in Y, we get

$$v_{23} - \bar{p}_{33} + \epsilon \geq v_{24} - \bar{p}_{24}.$$

Adding above two equations we get $\hat{p}_{33} - \bar{p}_{33} + 3\epsilon \geq \hat{p}_{24} - \bar{p}_{24}$. Earlier, we had shown that $\hat{p}_{24} - \bar{p}_{24} > \delta - 2\epsilon$. So we get $\hat{p}_{33} - \bar{p}_{33} > \delta - 5\epsilon$.

So in all cases we showed that there exists a producer $\gamma_3 \notin \mathcal{J}$ which is assigned an item α_3 to be consumed by $\beta_2 \notin \mathcal{I}$ in Y such that $\bar{p}_{33} < \hat{p}_{33} - \delta + 5\epsilon$. ■

To prove the theorem set $\mathcal{I} = \{\beta_1\}$, $\mathcal{J} = \{\gamma_1\}$ and $\delta = 5w\epsilon$. Applying lemma 1, we will discover a new consumer and a new producer, γ_2, which is assigned an item α_2 in Y such that $\hat{p}_{22} - \bar{p}_{22} > 5(w-1)\epsilon$. We can repeat this process until we exhaust either all the producers or consumers in the system at which point we will have $\hat{p}_{ww} - \bar{p}_{ww} > 5\epsilon$. But we can still apply lemma 4, to discover new pair of producer and consumer. This gives us a contradiction. ■

Proof **(Theorem 5):** Assume for contradiction that producer γ_1 does not produce any item in SIA. Since $s^{max}_{\gamma_1} - (2w+1)\epsilon \geq 0$, γ_1 has positive surplus in maximum competitive equilibrium. So he will be assigned in maximum competitive equilibrium. Let this assignment involve the tuple (β_1, α_1). As γ_1 follows an ISB strategy and does not produce any item in SIA, by proposition 4, $\hat{p}_{11} - c_{11} < \hat{s}_1 + \epsilon$. Also $p^{max}_{11} - c_{11} = s^{max}_{\gamma_1}$. From the above two equations we get, $p^{max}_{11} - \hat{p}_{11} > s^{max}_{\gamma_1} - \hat{s}_1 - \epsilon \geq 2w\epsilon$. Hence $p^{max}_{11} - \hat{p}_{11} > 2w\epsilon$. Since β_1 is assigned in maximum competitive equilibrium assignment and the final price of tuples (γ_1, α_1) in SIA is more than $2w\epsilon$

below the maximum competitive equilibrium price, she should have more than $2w\epsilon$ surplus on (γ_1, α_1) in SIA. By proposition 2 and LGB(0) strategy, β_1 should be assigned in SIA. Let her assignment be (γ_2, α_2) in SIA.

Proposition 5 *If β_1 gets (γ_1, α_1) (or has (γ_1, α_1) in her demand set) in any competitive equilibrium assignment and gets (γ_2, α_2) in SIA by following LGB strategy, such that $\hat{p}_{11} < p_{11} - \delta$ $(\delta \geq \epsilon)$, where $\boldsymbol{p}$ is the corresponding competitive equilibrium price vector then $\hat{p}_{22} < p_{22} - \delta + \epsilon$.*

Proof: From proposition 1, we have,

$$v_{12} - \hat{p}_{22} \geq v_{11} - \hat{p}_{11} - \epsilon.$$

From the property of competitive equilibrium we have,

$$v_{11} - p_{11} \geq v_{12} - p_{22}.$$

Adding above two equations we have,

$$p_{22} - \hat{p}_{22} + \epsilon \geq p_{11} - \hat{p}_{11}.$$

Since $\hat{p}_{11} < p_{11} - \delta$. This reduces the previous equation to $p_{22} - \hat{p}_{22} + \epsilon > \delta$. Thus $p_{22} - \hat{p}_{22} > \delta - \epsilon$. ■

From proposition 5, and the fact that $p_{11}^{max} - \hat{p}_{11} > 2w\epsilon$, we have $p_{22}^{max} - \hat{p}_{22} > (2w-1)\epsilon$. Since γ_2 produces an item in SIA, and $\hat{s}_{\gamma_2} \geq 0$, from the previous equation it can be concluded that γ_2 will have more than $(2w-1)\epsilon$ surplus in maximum competitive equilibrium. So, by the property of competitive equilibrium, he will be assigned some tuple, (β_2, α_3) in maximum competitive equilibrium.

Proposition 6 *If γ_2 produces α_2 in SIA by following an ISB strategy and α_3 (or has α_3 in his supply set) in a competitive equilibrium assignment, such that $p_{22} - \hat{p}_{22} > \delta$ $(\delta \geq \epsilon)$, where $\boldsymbol{p}$ is the competitive equilibrium price vector than $p_{23} - \hat{p}_{23} > \delta - \epsilon$.*

Proof: From the property of competitive equilibrium we have,

$$p_{23} - c_{23} \geq p_{22} - c_{22}.$$

From proposition 3, we have,

$$\hat{p}_{22} - c_{22} + \epsilon \geq \hat{p}_{23} - c_{23}$$

Adding previous two equations and using the fact that $p_{22} - \hat{p}_{22} > \delta$, we get $p_{23} - \hat{p}_{23} > \delta - \epsilon$. ■

From proposition 6, we have $p_{23}^{max} - \hat{p}_{23} > (2w-2)\epsilon$. Now as β_2 is assigned in maximum competitive equilibrium, the previous equation implies that she should have more than $2(w-2)\epsilon$ surplus in SIA. So she should be assigned to some tuple (γ_3, α_4) in SIA by proposition 2 and LGB(0) strategy. This process of discovering new producers and consumers can continue till we exhaust either of them. At that stage we will have $p_{wi}^{max} - \hat{p}_{wi} > 2\epsilon$, where α_i is any item in $\mathcal{A}$. But we can still apply the same logic and discover a new set of producer and consumer. But as $w = min(l, n)$, we have exhausted either all the producers or all the consumers in the system. So we reach a contradiction. Therefore γ_1 will produce some item in SIA. ■

Proof **(Theorem 6):** Let $\hat{\mathbf{p}}$ denote the final prices in SIA. Now consider the following important lemma.

Lemma 2 *Let $\mathcal{J}$ be a set of producers assigned in SIA such that if $\gamma_k \in \mathcal{J}$ is assigned to produce α_j then $\hat{p}_{kj} \geq p_{kj}^{max} + \delta$ ($\delta \geq 2\epsilon$). There exists a producer $\gamma_{k'} \notin \mathcal{J}$, assigned to $(\beta_{i'}, \alpha_{j'})$ in SIA such that $\hat{p}_{k'j'} \geq p_{k'j'}^{max} + \delta - 2\epsilon$.*

Proof: Let $\mathcal{I}$ be the set of consumers to which producers in $\mathcal{J}$ are assigned in SIA. As $\hat{p}_{kj} \geq p_{kj}^{max} + \delta$ and all consumers follow LGB(t) strategy with $t \geq 0$, all consumers in $\mathcal{I}$ will have at least $\delta \geq 2\epsilon$ surplus in maximum competitive equilibrium. By the property of competitive equilibrium, all consumers in $\mathcal{I}$ will be assigned in maximum competitive equilibrium. There are two cases to consider:

Case 1: A consumer in $\mathcal{I}$ is assigned a producer $\notin \mathcal{J}$ in maximum competitive equilibrium. Let that consumer be $\beta_i \in \mathcal{I}$ and her maximum competitive equilibrium assignment be $(\gamma_{k'}, \alpha_2)$ such that $\gamma_{k'} \notin \mathcal{J}$. Let the assignment of β_i in SIA be (γ_k, α_1) where $\gamma_k \in \mathcal{J}$.

Proposition 7 *If $\mathbf{p}$ is competitive equilibrium price vector and β_i is assigned (γ_k, α_1) in SIA with consumers following LGB strategies and (γ_k', α_2) is in β_i's demand set in the corresponding competitive equilibrium such that $\hat{p}_{k1} - p_{k1} \geq \delta$, then $\hat{p}_{k'2} - p_{k'2} \geq \delta - \epsilon$.*

Proof: From proposition 1, we have,

$$v_{i1} - \hat{p}_{k1} + \epsilon \geq v_{i2} - \hat{p}_{k2}.$$

From the property of competitive equilibrium and demand set we have

$$v_{i2} - p_{k'2} \geq v_{i1} - p_{k'1}.$$

Adding these two equations we get $\hat{p}_{k'2} - p_{k'2} + \epsilon \geq \hat{p}_{k1} - p_{k1} \geq \delta$. Hence $\hat{p}_{k'2} - p_{k'2} \geq \delta - \epsilon$. ■

By proposition 7 we have $\hat{p}_{k'2} - p_{k'2}^{max} \geq \delta - \epsilon$. Consider the following proposition.

Proposition 8 *If a producer γ_1 follows ISB($\hat{s}_1$) strategy such that $\hat{s}_1 \leq s_1^{max}$ and $\hat{p}_{1i} \geq p_{1i}^{max} + \delta$ ($\delta \geq \epsilon$) for some item $\alpha_i \in \mathcal{A}$ which is in the supply set of γ_1, then γ_1 will produce some item in SIA.*

Proof: Since $\hat{p}_{1i} \geq p_{1i}^{max} + \delta$, this means $\hat{p}_{1i} - c_{1i} \geq p_{1i}^{max} - c_{1i} + \delta = s_1^{max} + \delta \geq \hat{s}_1 + \delta$. This means $\hat{p}_{1i} - c_{1i} \geq \hat{s}_1 + \delta$. By proposition 4, γ_1 will be assigned in SIA. ■

By proposition 8, $\gamma_{k'}$ will be assigned in SIA. Let that assignment involve item α_3.

Proposition 9 *If $\boldsymbol{p}$ is a competitive equilibrium price vector and $\gamma_{k'}$ produces α_3 in SIA by following an ISB strategy but α_2 is in his supply set in competitive equilibrium such that $\hat{p}_{k'2} - p_{k'2} \geq \delta$, then $\hat{p}_{k'3} - p_{k'3} \geq \delta - \epsilon$.*

Proof: From proposition 3, we have,

$$\hat{p}_{k'3} - c_{k'3} + \epsilon \geq \hat{p}_{k'2} - c_{k'2}.$$

From the property of competitive equilibrium we have,

$$p_{k'2} - c_{k'2} \geq p_{k'3} - c_{k'3}.$$

Adding the above two equations we get $\hat{p}_{k'3} - p_{k'3} + \epsilon \geq \hat{p}_{k'2} - p_{k'2} \geq \delta$. This gives us $\hat{p}_{k'3} - p_{k'3} \geq \delta - \epsilon$. ■

By proposition 9, we have $\hat{p}_{k'3} - p_{k'3}^{max} \geq \delta - 2\epsilon$.

Case 2: All the consumers in $\mathcal{I}$ are assigned producers $\in \mathcal{J}$ in maximum competitive equilibrium. Consider the following proposition:

Proposition 10 *Consider a set $\mathcal{Q}$ of $(producer, item)$ tuples which are assigned to a set $\mathcal{R}$ of consumers in maximum competitive equilibrium who have positive surplus on them. There exists (γ_1, α_1) tuple $\notin \mathcal{Q}$ which is in the demand set of one of the consumers $\in \mathcal{R}$ and supply set of γ_1 includes α_1.*

Proof: Assume for contradiction that all the consumers $\in \mathcal{R}$ have their $(producer, item)$ demand set tuples from $\mathcal{Q}$. Since all consumers in $\mathcal{R}$ have positive surplus, we can increase the prices of all the tuples in $\mathcal{Q}$ by the same sufficiently small amount without changing the demand set of consumers in $\mathcal{R}$. Since the price of an item is increased, that item still remains in the supply set of its corresponding producer for all tuples $(\gamma_k, \alpha_j) \in \mathcal{Q}$. Therefore, we still have competitive equilibrium, with increased price. But we were already at the maximum competitive equilibrium. So our assumption is incorrect. Hence, there exists a $(producer, item)$ tuple $\notin \mathcal{Q}$ which is in the demand set of one of the consumers $\in \mathcal{R}$. Let this tuple be (γ_1, α_1).

Now assume for contradiction that γ_1 does not have α_1 in his supply set. This means no consumer is assigned to (γ_1, α_1) in maximum competitive equilibrium. So we can increase the price of (γ_1, α_1) by sufficiently small amount without changing the supply set of γ_1 and still have competitive equilibrium. This is a contradiction as we are already at maximum competitive equilibrium. ■

Applying proposition 10, we get that there exists a tuple (γ_2, α_2) such that $\gamma_2 \notin \mathcal{J}$ and $\beta_1 \in \mathcal{I}$ has (γ_2, α_2) in her demand set. Let the assignment of β_1 in SIA be (γ_1, α_1). From proposition 7, we have $\hat{p}_{22} - p_{22}^{max} \geq \delta - \epsilon$. Also, from proposition 10, α_2 is in the supply set of γ_2. From proposition 8, γ_2 will be assigned in SIA. Let that assignment involve item α_3. So we have from proposition 9, $\hat{p}_{23} - p_{23}^{max} \geq \delta - 2\epsilon$.
This proves the lemma in both the possible cases. ■

The proof of the theorem can be done by repeatedly applying Lemma 2. Assume γ_1 will produce an item α_1 in SIA. Set $\mathcal{J} = \gamma_1$. Since $\hat{s}_1 \geq s_{\gamma_1}^{max} + 2w\epsilon$, we have $\hat{p}_{11} \geq p_{11}^{max} + 2w\epsilon$. Set $\delta = 2w\epsilon$ and apply Lemma 2 to get another tuple (γ_2, α_2) assigned to another consumer β_2 such that $\hat{p}_{22} - p_{22}^{max} \geq 2w\epsilon - 2\epsilon$. By proposition 8, γ_2 will be assigned in SIA. Again we can apply Lemma 2 by including γ_2 in $\mathcal{I}$ and setting $\delta = 2w\epsilon - 2\epsilon$ to discover another pair of producer and consumer. This process can continue till we either run out of producers or consumers in the system. At that moment, we will have $\hat{p}_{ww} - p_{ww}^{max} \geq 2\epsilon$. But we can still apply lemma 2 and discover new producer and consumer. This gives us a contradiction. ■

Proof **(Theorem 7):** $\mathbf{p}^{max}$ is a competitive equilibrium price vector and $\mathbf{s}(\mathbf{p}^{max}) = \mathbf{s}_p^{max} \geq \mathbf{s}$. This implies that, there exists a minimal competitive equilibrium price vector at $\mathbf{s}$.

Assume for contradiction, $\mathbf{p}^1$ and $\mathbf{p}^2$ are two different ($\mathbf{p}^1 \neq \mathbf{p}^2$) minimal competitive equilibrium price vectors at $\mathbf{s}$. Let $\mathbf{p}^3$ be a price vector such that $p_{ki}^3 = min[p_{ki}^1, p_{ki}^2]\ \forall \gamma_k \in \mathcal{G},\ \forall \alpha_i \in \mathcal{A}$. Define $s_{\gamma_k}^3 = min[s_{\gamma_k}(\mathbf{p}^1), s_{\gamma_k}(\mathbf{p}^2)]$.

From the lattice property of the competitive equilibrium price vector space, $\mathbf{p}^3$ is a competitive equilibrium price vector with $\mathbf{s}(\mathbf{p}^3) = \mathbf{s}^3$ (Gul and Stacchetti, 1999). Since $\mathbf{p}^1$ and $\mathbf{p}^2$ are different price vectors, from definition of $\mathbf{p}^3$, we have $\mathbf{p}^3 < \mathbf{p}^1$ and $\mathbf{p}^3 < \mathbf{p}^2$. Since $\mathbf{s}(\mathbf{p}^1) \geq \mathbf{s}$ and $\mathbf{s}(\mathbf{p}^2) \geq \mathbf{s}$, $\mathbf{s}^3 \geq \mathbf{s}$. As $\mathbf{s}^3 = \mathbf{s}(\mathbf{p}^3)$, $\mathbf{s}(\mathbf{p}^3) \geq \mathbf{s}$. So, $\mathbf{p}^3$ is a minimal competitive equilibrium price vector and $\mathbf{p}^1$, $\mathbf{p}^2$ are not. This is a contradiction. Therefore, the minimal competitive equilibrium price vector at $\mathbf{s}$ is unique. ■

Proof **(Theorem 8):** Let us denote the minimum competitive equilibrium corresponding to $\hat{\mathbf{s}}$ as μ. Let us denote the corresponding price vector as $\mathbf{p}$. Now consider the following lemma.

Lemma 3 *Let $\mathcal{J}$ be a set of producers who are assigned in μ, such that $\forall \gamma_k \in \mathcal{J}$ and for some $\alpha_i \in S_k(\boldsymbol{p}), \hat{p}_{ki} < p_{ki} - \delta(\delta \geq 2\epsilon)$. Then there exists a producer $\gamma_1 \notin \mathcal{J}$, such that (γ_1, α_3) is assigned some consumer in μ and $\hat{p}_{13} < p_{13} - \delta + 2\epsilon$.*

Proof: Let $\mathcal{I}$ be the set of consumers who are assigned to $\mathcal{J}$ in μ. Since $\hat{p}_{ki} < p_{ki} - \delta$, consumers from $\mathcal{I}$ will have at least δ surplus in SIA. Since all the consumers follow the LGB(0) strategy, proposition 2 implies that they should be assigned in SIA. There are two possible cases.

Case 1: Some consumer in $\mathcal{I}$ is assigned a producer $\notin \mathcal{J}$ in SIA. WLOG, let that consumer be β_1. Let her assignment in SIA and μ be (γ_1, α_1) and (γ_2, α_2) respectively. From proposition 5, we have $p_{11} - \hat{p}_{11} > \delta - \epsilon$. Since γ_1 is assigned in SIA and $p_{11} - \hat{p}_{11} > \delta - \epsilon$, he should have at least $\hat{s}_{\gamma_1} + \delta - \epsilon$ surplus in μ. Since $\hat{s}_{\gamma_1} \geq 0$, γ_1 should be assigned in μ. Let one of the items in his supply set be α_3. By proposition 6, we have $p_{13} - \hat{p}_{13} > \delta - 2\epsilon$.

Case 2: All consumers in $\mathcal{I}$ get producers from $\mathcal{J}$ in SIA. Consider the following proposition:

Proposition 11 *A consumer $\beta_1 \notin \mathcal{J}$ will have a (producer,item) tuple (say (γ_2, α_2)) in her demand set such that $\gamma_2 \in \mathcal{I}$ and α_2 is in the supply set of producer γ_2.*

Proof: Consider the tuple (γ_k, α_i) such that $\gamma_k \in \mathcal{J}$ and $\alpha_i \in S_k(\mathbf{p}$. Since $p_{ki} - \hat{p}_{ki} > \delta$ and γ_k is assigned in SIA using ISB($\hat{s}_k$) strategy, then γ_k should have at least $\hat{s}_k + \delta$ surplus in μ. So all the producers in $\mathcal{J}$ will have $\hat{\mathbf{s}} + \delta$ surplus in μ. Assume for contradiction that the proposition is not true. Decrease the prices of all tuples (γ_k, α_i) such that $\gamma_k \in \mathcal{I}$ and α_i is in the supply set of γ_k, by a sufficiently small amount. By decreasing the prices like this, the supply set of producers in $\mathcal{J}$ remains unchanged. The demand set of all consumers $\beta_1 \notin \mathcal{I}$ does not contain a tuple (γ_2, α_2) such that $\gamma_2 \in \mathcal{J}$ and α_2 is in the supply set of γ_2. Therefore the demand set of all such consumers remains unchanged. The demand set of all the consumers in $\mathcal{I}$ would still contain the tuple assigned to them, since the prices decreased for all such tuples. Therefore, the new vector of prices ($\mathbf{p}'$) is still a competitive equilibrium price (with the same competitive equilibrium assignment), and $s(\mathbf{p}') \geq \hat{\mathbf{s}}$. Therefore, $\mathbf{p}$ was not the minimum competitive equilibrium price vector at $\hat{\mathbf{s}}$. This is a contradiction to our assumption that the proposition is not true. ■

From the above proposition, $\beta_1 \notin \mathcal{I}$ will have (γ_2, α_2) in her demand set in μ such that $\gamma_2 \in \mathcal{J}$. Since the prices in SIA are below the prices in a competitive equilibrium by at least δ, β_1 should have at least δ surplus on tuple (γ_2, α_2) in SIA. Since β_1 follows the LGB(0) strategy, using proposition 2, β_1 will be assigned some (γ_1, α_1) in SIA. Since $\beta_1 \notin \mathcal{I}, \gamma_1 \notin \mathcal{J}$. From proposition 5 we have $p_{11} - \hat{p}_{11} > \delta - \epsilon$. Since γ_1 is assigned in SIA and $p_{11} - \hat{p}_{11} > \delta - \epsilon$, he should have at least $\hat{s}_{\gamma_1} + \delta - \epsilon$ surplus in μ. Since $\hat{s}_{\gamma_1} \geq 0$, γ_1 should be assigned in μ. Let one of the items in his supply set be α_3. By proposition 6, we get $p_{13} - \hat{p}_{13} > \delta - 2\epsilon$.

So in both cases we proved the lemma. ■

To prove the theorem, we subdivide all the $(producer, item)$ tuples into the following two categories.

Case 1: Consider the tuples (γ_1, α_1), that are assigned to a consumer, say β_1 in μ. Assume for contradiction that $p_{11} - \hat{p}_{11} > 2w\epsilon$. Apply lemma 3 with $\mathcal{I} = \{\gamma_1\}, \delta = 2w\epsilon$, to discover a new producer γ_2 which is assigned an item α_2 to be produced for a new consumer β_2 such that $p_{22} - \hat{p}_{22} > 2w\epsilon - 2\epsilon$. We can apply lemma 3 again to discover another pair of producer and consumer which are matched in μ. This process will continue till we run out of producers or consumers in the system at which point we will have $p_{ww} - \hat{p}_{ww} > 2\epsilon$. But we can still apply lemma 3 and discover a new pair of producer and consumer. This gives us a contradiction. So the theorem holds. So $p_{11} - \hat{p}_{11} \leq 2w\epsilon$.

Case 2: Consider the tuple, (γ_1, α_1) such that γ_1 is not assigned α_1 in μ. There will be a consumer, β_1, which will have the tuple (γ_1, α_1) in her demand set in μ. If this is not true, then we can decrease the price of this tuple and still have a minimal competitive equilibrium at $\hat{\mathbf{s}}$. This implies $v_{11} - p_{11} \geq 0$. Assume for contradiction, $p_{11} - \hat{p}_{11} > 2(w+1)\epsilon$. This gives us, $v_{11} - \hat{p}_{11} > 2(w+1)\epsilon$. By LGB(0) strategy and proposition 2, β_1 will be assigned in SIA. Let that assignment involve (γ_2, α_2). So applying proposition 5, we get $p_{22} - \hat{p}_{22} > (2w+1)\epsilon$. As $\hat{p}_{22} - c_{22} \geq \hat{s}_2$, we can write, $p_{22} - c_{22} > \hat{s}_2 + (2w+1)\epsilon$. Since $\hat{s}_2 \geq 0$, by the property of competitive equilibrium, γ_2 will be assigned some item in μ. Let that item be α_3. By proposition 6, we get $p_{23} - \hat{p}_{23} > 2w\epsilon$. But in case 1 of the proof, we showed that $p_{23} - \hat{p}_{23} \leq 2w\epsilon$. This gives us a contradiction.

So we conclude that $\hat{\mathbf{p}} \geq \mathbf{p}^{min}(\mathbf{s}) - 2(w+1)\epsilon$. ■

Proof **(Theorem 9):** Let us denote the minimum competitive equilibrium price vector at $\hat{\mathbf{s}}$ as $\mathbf{p}$. Now, consider the following important lemma.

Lemma 4 *Let $\mathcal{J}$ be a set of producers who are assigned in SIA such that $\hat{p}_{kj} > p_{kj} + \delta$ ($\delta \geq 4\epsilon$), where $\gamma_k \in \mathcal{J}$ and is assigned item α_j in SIA. Then there exists a producer $\gamma'_k \notin \mathcal{J}$ which is assigned an item α'_j such that $\hat{p}_{k'j'} > p_{k'j'} + \delta - 4\epsilon$.*

Proof: Let $\mathcal{I}$ be the set of consumers assigned to producers in $\mathcal{J}$ in SIA. Since prices of producers in $\mathcal{J}$ in SIA are above the competitive equilibrium price by δ, then all the consumers in $\mathcal{I}$ will have at least δ surplus in competitive equilibrium. So all the consumers in $\mathcal{I}$ will be assigned in competitive equilibrium. There are two cases to consider.

Case 1: There exists a consumer, $\beta_1 \in \mathcal{I}$ whose competitive equilibrium assignment is (γ_1, α_1) and $\gamma_1 \notin \mathcal{I}$. Let the assignment of β_1 in SIA be (γ_2, α_2). From proposition 7, we have $\hat{p}_{11} - p_{11} > \delta - \epsilon$. By property of minimum competitive equilibrium at $\hat{\mathbf{s}}$, we get $\hat{p}_{11} > p_{11} + \delta - \epsilon \geq c_{11} + \hat{s}_1 + \delta - \epsilon$. As $\hat{s}_1 \geq 0$ and $\delta \geq 4\epsilon$, applying proposition 4, γ_1 should be assigned an item in SIA. Let that be α_3. From proposition 9 we have $\hat{p}_{13} > p_{13} + \delta - 2\epsilon$.
Case 2: All consumers in $\mathcal{J}$ get producers from $\mathcal{I}$ in competitive equilibrium. Let β_1 be the consumer who placed a *preferred bid* on one of the items produced by producers in $\mathcal{J}$, and her bid was the last such bid that was superseded. Easy to see that $\beta_1 \notin \mathcal{I}$. Let $\gamma_1 \in \mathcal{J}$ be the corresponding producer. Observe that after β_1 has been outbid, no other preferred bid will be placed on items produced by producers of $\mathcal{J}$. So β_1 is either outbid by a consumer or the producer agent of γ_1. We consider both the cases.
Case 2(a): β_1 is outbid by another consumer β_2. Let the assigned item of γ_1 in SIA be α_1. This means β_1 must be the highest bidder when the price of (γ_1, α_1) was $\hat{p}_{11} - \epsilon$. So $v_{11} - \hat{p}_{11} + \epsilon \geq 0$. But we have $\hat{p}_{11} - p_{11} > \delta \geq 4\epsilon$. So $v_{11} - p_{11} > 3\epsilon$. This means β_1 will be assigned something in competitive equilibrium. Let it be (γ_2, α_2). As $\beta_1 \notin \mathcal{I}$, $\gamma_2 \notin \mathcal{J}$. The price of (γ_1, α_1), when β_1 bid for the last time on it must be $\hat{p}_{11} - 2\epsilon$. Let the price of (γ_2, α_2) at this instant be $\bar{p}_{22}$. From the property of LGB strategy,

$$v_{11} - \hat{p}_{11} + 2\epsilon \geq v_{12} - \bar{p}_{22} \geq v_{12} - \hat{p}_{22}.$$

From the property of competitive equilibrium we have

$$v_{12} - p_{22} \geq v_{11} - p_{11}.$$

Adding the above two equations and using the fact that $\hat{p}_{11} - p_{11} > \delta$, we have $\hat{p}_{22} - p_{22} + 2\epsilon \geq \hat{p}_{11} - p_{11} > \delta$. This gives us $\hat{p}_{22} - p_{22} > \delta - 2\epsilon$. By property of minimal competitive equilibrium at $\hat{\mathbf{s}}$, we have $\hat{p}_{22} > p_{22} + \delta - 2\epsilon \geq c_{22} + \hat{s}_2 + \delta - 2\epsilon$. As $\hat{s}_2 \geq 0$ and $\delta \geq 4\epsilon$, by proposition 4, γ_2 should be assigned in SIA. Let that assignment involve α_3. From proposition 9 we have, $\hat{p}_{23} > p_{23} + \delta - 3\epsilon$.
Case 2(b): β_1 gets outbid by the producer agent of γ_1. Let the item on which β_1 gets outbid be α_3. Let the assignment of γ_1 in competitive equilibrium involve α_1. Consider the instant in SIA when β_1 had the last outstanding bid on (γ_1, α_3). The price of (γ_1, α_3) at this instant was $\hat{p}_{13} - \epsilon$. Since β_1 placed a preferred bid on (γ_1, α_3) and was the last such consumer to get outbid, the surplus of γ_1 can increase atmost by ϵ later in SIA. Let s_{γ_1} be the surplus of γ_1 at the end of SIA. So we can say by proposition 3,

$$(\hat{p}_{13} - \epsilon) - c_{13} + \epsilon \geq s_{\gamma_1} \geq \hat{p}_{11} - c_{11} - \epsilon.$$

By property of competitive equilibrium we have,

$$p_{11} - c_{11} \geq p_{13} - c_{13}.$$

Adding the previous two equations we get, $\hat{p}_{13} - p_{13} + \epsilon \geq \hat{p}_{11} - p_{11} > \delta$. So $\hat{p}_{13} - p_{13} > \delta - \epsilon$. Since β_1 placed a bid on (γ_1, α_3) at the price $\hat{p}_{13} - 2\epsilon$, $v_{13} - \hat{p}_{13} + \epsilon \geq 0$. This means $v_{13} - p_{13} > \delta - 2\epsilon \geq 2\epsilon$. Hence β_1 should be assigned in the competitive equilibrium. Let her assignment involve (γ_2, α_2). As $\beta_1 \notin \mathcal{I}$, $\gamma_2 \notin \mathcal{J}$. Let the price of (γ_2, α_2), when β_1 was about to place her last bid on (γ_1, α_3), be $\bar{p}_{22}$. By ISB strategy and the fact that prices do not decrease in SIA, we have

$$v_{13} - \hat{p}_{13} + 2\epsilon \geq v_{12} - \bar{p}_{22} \geq v_{12} - \hat{p}_{22}.$$

By property of competitive equilibrium we have,

$$v_{12} - p_{22} \geq v_{13} - p_{13}.$$

Adding the previous two equations we get $\hat{p}_{22} - p_{22} + 2\epsilon \geq \hat{p}_{13} - p_{13} > \delta - \epsilon$. Hence $\hat{p}_{22} - p_{22} > \delta - 3\epsilon$. By the property of minimum competitive equilibrium at $\hat{\mathbf{s}}$, we get $\hat{p}_{22} > p_{22} + \delta - 3\epsilon \geq c_{22} + \hat{s}_2 + \delta - 3\epsilon$. As $\hat{s}_2 \geq 0$ and $\delta \geq 4\epsilon$, by proposition 4, γ_2 should be assigned in SIA. Let that assignment involve α_4. From proposition 9, we get, $\hat{p}_{24} - p_{24} > \delta - 4\epsilon$. ■

To prove the theorem, we explore two possible cases.

Case 1: γ_1 is assigned α_1 in SIA. Assume for contradiction $\hat{p}_{11} - p_{11} > (4w+3)\epsilon$. So we have $\hat{p}_{11} - p_{11} > 4w\epsilon$. Set $\mathcal{I} = \{\gamma_1\}$ and $\delta = 4w\epsilon$. Applying lemma 4, we will discover a new consumer and a new producer, γ_2, which is assigned an item α_2 in SIA such that $\hat{p}_{22} - p_{22} > 4(w-1)\epsilon$. We can repeat this process until we exhaust either all the producers or consumers in the system at which point we will have $\hat{p}_{ww} - p_{ww} > 4\epsilon$. But we can still apply lemma 4, to discover new pair of producer and consumer. This gives us a contradiction. This shows that if γ_1 is assigned α_1 in SIA, then $\hat{p}_{11} - p_{11} \leq 4w\epsilon$.

Case 2: γ_1 is not assigned α_1 in SIA. If no consumer places bid on this tuple then $\hat{p}_{11} = 0$ and the proof is trivial. Else, a consumer, β_2, should have placed a bid on (γ_1, α_1), when its price was $\hat{p}_{11} - 2\epsilon$. Assume for contradiction, $\hat{p}_{11} - p_{11} > (4w+3)\epsilon$. By LGB(0) strategy, $v_{21} - \hat{p}_{11} + \epsilon \geq 0$. This means $v_{21} - p_{11} > (4w+2)\epsilon$. Hence β_2 should be assigned in the competitive equilibrium. Let this assignment involve (γ_2, α_2). Let the price of (γ_2, α_2) when β_2 places her last bid on (γ_1, α_1), be $\bar{p}_{22}$. By property of LGB strategy and the fact that prices in SIA do not decrease, we have

$$v_{21} - \hat{p}_{11} + 2\epsilon \geq v_{22} - \bar{p}_{22} \geq v_{22} - \hat{p}_{22}.$$

By property of competitive equilibrium we have

$$v_{22} - p_{22} \geq v_{21} - p_{11}.$$

Adding the previous two equations we get, $\hat{p}_{22} - p_{22} + 2\epsilon \geq \hat{p}_{11} - p_{11} > (4w+3)\epsilon$. Hence $\hat{p}_{22} - p_{22} > (4w+1)\epsilon$. By the property of minimum competitive equilibrium at $\hat{\mathbf{s}}$, we get $\hat{p}_{22} > p_{22} + (4w+1)\epsilon \geq c_{22} + \hat{s}_2 + (4w+1)\epsilon$. As $\hat{s}_2 \geq 0$, by proposition 4, γ_2 should be assigned in SIA. Let that assignment involve α_3. From proposition 9, we get, $\hat{p}_{23} - p_{23} > 4w\epsilon$. But we have already shown that if γ_2 gets assigned to α_3 in SIA, then $\hat{p}_{23} - p_{23} \leq 4w\epsilon$. So we reach a contradiction.

Since we reach contradictions in both cases, $\hat{p}_{11} - p_{11} \leq (4w+3)\epsilon$. ■