COL758: Advanced Algorithms	**Spring 2019**

Lecture 9: January 31

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9.1 Electrical Flow

The idea of current flowing through electrical network of resistors can be used to solve maxflow problem. Electrical network are suitable to model maxflow because it has properties that are constraints in the maxflow 1. conservation of charge at a node which translate to net flow going inside a node equals the net flow out form the node.

current through resistor between node u and v

$$I(u,v) = \frac{\pi_u - \pi_v}{R_{uv}}$$

where π_u and π_v is potential of nodes u and v respectively

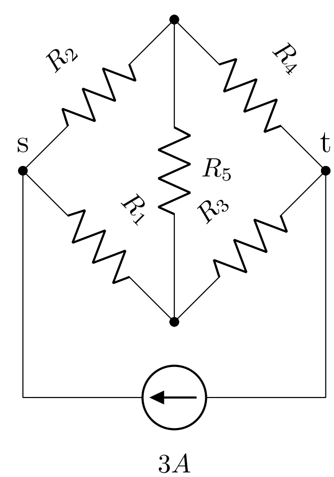

Figure 9.1: Electrical Network

Conservation of Current

Net flow of current (sum of current coming into a node = sum of current going out of it) out of a node is zero, which physically means a node can't hold charge.

$$\forall nodes v \quad \sum_{u:(v,u)\in E} (\pi_v - \pi_u)c_{vu} = 0$$

with exception for source(s) and sink(t):

$$\sum_{u:(s,u)\in E} (\pi_s - \pi_u)c_{su} = 3A$$

$$\sum_{u:(t,u)\in E} (\pi_t - \pi_u)c_{tu} = -3A$$

here conductance $c_{vu} = \frac{1}{R_{vu}}$

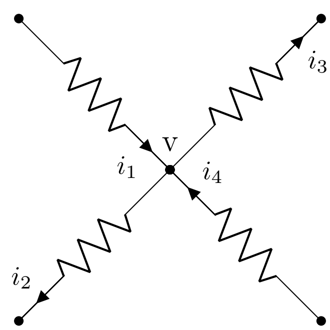

Figure 9.2: Current at a node ($-i_1 + i_2 + i_3 - i_4 = 0$; say, outward is positive)

The above set of equation can be written in matrix form

$$\underbrace{\begin{bmatrix} \sum_{u:(1,u)\in E} c_{1u} & \cdots & \cdots & \cdots & -c_{1t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \sum_{u:(v,u)\in E} c_{vu} & \cdots & -c_{vt} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -c_{s1} & \cdots & \cdots & \cdots & -c_{st} \\ -c_{t1} & \cdots & \cdots & \cdots & \sum_{u:(t,u)\in E} c_{tu} \end{bmatrix}}_{\text{conductance}} \underbrace{\begin{bmatrix} \pi_1 \\ \vdots \\ \pi_v \\ \vdots \\ \pi_s \\ \pi_t \end{bmatrix}}_{\text{potential}} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 3 \\ -3 \end{bmatrix}}_{\text{current out of a node}} \tag{9.1}$$

9.2 Graph Laplacian

Graph Laplacian is a symmetric matrix, it can be obtained for any graph given its adjacency matrix. Graph Laplacian is studied in graph spectral theory.

Adjacency Matrix of a Graph, $G(V, E)$

Adjacency matrix of an un-directed graph is a symmetric matrix of $n \times n(n = \#nodes)$ of values 0 or 1. If there is an edge between two nodes the corresponding entry is 1, otherwise 0. Diagonal entries are zero, because there are no self loop in the graph(say).

The adjacency matrix for the graph of Figure 9.3, is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

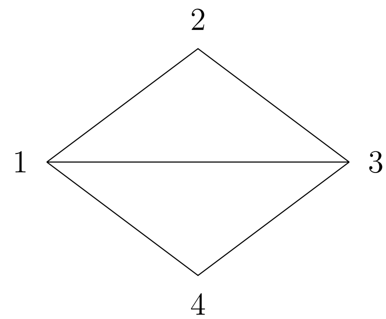

Figure 9.3: Undirected Graph with 4 nodes and 5 edges

The degree of a node equals number of edges incident upon it. The matrix **D** for a graphs is a diagonal matrix with diagonal entries showing degree of a node. **D** can also be obtained by adding all the entries of a row of **A**

D for the figure 9.3, is

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Graph Laplacian for a graph G is defined as

$$L(G) = D(G) - A(G)$$

For graphs, figure 9.3

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Electrical flow as Weighted Laplacian

The electrical flow conductance matrix is an example of weighted Laplacian.

$$\text{Where} \qquad A_{ij} = c_{ij}$$

$$\text{and} \qquad D_{ii} = \sum_{j:(i,j)\in E} c_{i,j}$$

The system of equation 9.1 can be written using laplacian as

$$L\pi = b$$

This is interesting because Laplacian can be solved in linear time algorithm, which is linear in number of non-zero entries.