COL758: Advanced Algorithms **Spring 2019**

Lecture 8: January 28

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8.1 Integral LP for Set Cover

In last class, we have seen what an integral linear program for set cover problem. For Primal

$$\min \quad \sum_{S} C_j x_j$$

$$\text{s.t.}$$

$$\forall e_i \in U : \sum_{j:e_i \in S_j} x_j \geq 1$$

$$x_j \geq 0$$

For the Dual

$$\max \quad \sum_{i} y_i$$

$$\text{s.t.}$$

$$\forall S_j \sum_{i:e_i \in S_j} y_i \leq C_j$$

$$y_i \geq 0$$

8.2 Fractional Solution

We give requirements to each elements. Initially its value is 1.
At each step pick the set which minimizes

$$\min \quad \frac{C_j}{\sum_{e_i \in S_j} r_i}$$

Say the picked set is denoted with j then do

$$x_j \leftarrow x_j + \epsilon$$

$$\forall e_i \in S_j : y_i \leftarrow y_i + \rho_j r_i [1 - \frac{1}{1+\epsilon}]$$

$$r_i \leftarrow \frac{r_i}{1+\epsilon}$$

where $\rho_j = \frac{C_j}{\sum_{e_i \in S_j} r_i}$

For termination, we put constraint on r_i such that if $r_i < \delta$ then drop e_i from further consideration.
We will repeat above step until there is no element remaining for consideration.

Analysis

For every element, initially $r_i = 1$ and finally $r_i < \delta$

Every step decrease in r_i is by a factor of $\frac{1}{1+\epsilon}$

This implies that every element e_i is at least covered $\log_{1+\epsilon} \frac{1}{\delta}$

If an element e_i is covered then one of the sets containing it has its x value increase by ϵ

$$\sum_{e_i \in S_j} x_j \geq \epsilon \log_{1+\epsilon} \frac{1}{\delta}$$

$$X_j = \frac{\sum_{e_i \in S_j} x_j}{\log_{1+\epsilon} \frac{1}{\delta}} \geq 1$$

X_j is a feasible solution for primal LP.
In any step,

$$\text{increase in} \sum C_j x_j = C_j \epsilon$$

$$\text{increase in} \sum_i y_i = \frac{\epsilon}{1+\epsilon} C_j$$

At each step

$$(1+\epsilon)(\text{increase in} \sum_i y_i) = \text{increase in} \sum_i C_j x_j$$

For any set picked the ρ of that set is minimum. The increase in y_i for any element e_i is ρ times decrease in r_i. At each step,

$$\text{increase in} \sum y_i \leq \text{decrease in} \sum r_j \rho_j \text{ of } S_j$$

Consider any set S_j, initially

$$\sum_{e_i \in S_j} r_i = |S_j|$$

and finally it is at least δ.
$\sum_{e_i \in S_j} y_i$ is at most the total area under curve $\frac{C_j}{x}$ between δ and $|S_j|$.

$$\sum_{e_i \in S_j} y_i \leq C_j \log \frac{|S_j|}{\delta}$$

This is not a feasible solution for Dual. Let's scale it. Assume n is total number of elements in S_j.

$$Y_j = \frac{y_i}{\log \frac{n}{\delta}} \text{is a feasible solution}$$

8.3 Relation between the primal and dual solution

$$\begin{aligned}
\frac{\sum C_j X_j}{\sum Y_j} &= \frac{\sum C_j x_j}{\epsilon \log_{1+\epsilon} \frac{1}{\delta}} * \frac{\log \frac{n}{\delta}}{\sum y_i} \\
&= \frac{(1+\epsilon) \log \frac{n}{\delta}}{\epsilon \log_{1+\epsilon} \frac{1}{\delta}} \\
&= \frac{(1+\epsilon) \log \frac{n}{\delta}}{\epsilon \frac{\log \frac{1}{\delta}}{\log(1+\epsilon)}}
\end{aligned}$$

for small ϵ

$$= \frac{(1+\epsilon)\log\frac{n}{\delta}}{\log\frac{1}{\delta}}$$

Assume,

$$\delta = \frac{1}{n^{\frac{1}{\epsilon}}}$$

$$= (1+\epsilon) * \frac{\log n^{1+\frac{1}{\epsilon}}}{\log n^{\frac{1}{\epsilon}}}$$

$$= (1+\epsilon)^2$$

For small ϵ, the solution for primal and dual LP is very near the optimal solution.

8.3.1 Running Time

Each elements is covered

$$\log_{1+\epsilon}\frac{1}{\delta} = \frac{1}{\epsilon^2}\log n$$

times. So total number of steps for n elements will be

$$\frac{n}{\epsilon^2} * \log n$$

NOTE : More small the ϵ is, more better the solution will be but the time of execution will be more too.