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| **COL758: Advanced Algorithms** | **Spring 2019** |

# Lecture 5: January 14

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## 5.1 Max Flow Problem as a Linear Programming Problem

Given a graph $G = (V, E)$ with edge capacities $c_e : E \to R^+$ and a source $s$ and a sink $t$, we want to find the maximum flow such that the capacity constraints at each edge are satisfied. In previous classes we have seen how to formulate the max-flow problem as a linear program. Following is the linear program.
The primal can be formulated as:

$$\begin{aligned} \max \quad & \sum_{i:P_i \in ¶} f_i \\ \text{s.t.} \quad \forall e \in E \quad & \sum_{i:e \in P_i} f_i \le c_e \\ \forall i \quad & f_i \ge 0 \end{aligned}$$

where ¶: set of all paths from $s$ to $t$, $P_i \in$ ¶, $c_e$: capacity of each edge, $f_i$: flow along $P_i$.

Following is the dual of the above primal:

$$\begin{aligned} \max \quad & \sum_{e \in E} l_e c_e \\ \text{s.t.} \quad \forall P_i \in ¶ \quad & \sum_{e \in P_i} l_e \ge 1 \\ \forall e \in E \quad & l_e \ge 0 \end{aligned}$$

The above dual can be interpreted as choosing $l : E \to R^+$ s.t. the shortest path from $s$ to $t$ is at least 1 and $\sum_{e \in E} l_e c_e$ is minimum. We propose an iterative algorithm to solve the problem.

## 5.2 Algorithm

1. Assign a length $l_e$ with every edge $e$. Initialize $l_e$ as : $l_e^0 = \frac{\delta}{c_e}$ where $\delta$ is an input parameter. Also set flow through each edge = 0.
2. Repeat :
   (a) At each step we find shortest path(under current $l$ i.e. $l^{i-1}$ ) from $s$ to $t$, say $P^i$ and send $\lambda^{i-1} = \min_{e \in P^i} c_e$ units of flow along the path. For each edge on the path add this flow to their current flow values.
   (b) Updation step:- $\forall e \in P_i \quad l_e^i = l_e^{i-1}.(1 + \frac{\epsilon \lambda^{i-1}}{c_e})$ (more the edge is used, more the length of edge is increased.)

   until $D^i = 1$ where $D^i = \sum_e l_e^i c_e$

## Motivation

In this algorithm for each step we are getting different length functions that can be easily converted to feasible solutions of the dual problem by just scaling that is by dividing $D^i$ in each iteration with the length of the shortest path(let it be $\alpha^i$). So the new objective function will be $\frac{D^i}{\alpha^i}$ .
Let $\beta = \min_i \frac{D^i}{\alpha_i}$.
Now to calculate flow, we will keep track of how much flow we are sending along each edge and keep on adding in each iteration. But it is quite possible that we might send more flow than the capacity. For this, we will argue that if we divide flow of each edge by some fixed quantity, it will start following the capacity constraint and we will also argue that this primal solution will be close to $\beta$. We know from **weak duality theorem** that value of objective of dual is always greater than the objective value of primal for any feasible solution of the dual and primal respectively. Therefore if we get the objective values of primal and dual close to each other, then it must be close to the optimal solution as well.

## Analysis

For every iteration $i \geq 1$, the value of objective of dual which is $D^i$ is:

$$D^i = \sum_e l_e^i c_e$$

Substituting $l_e^i$ from the updation step we get,

$$\begin{aligned} D^i &= D^{i-1} + \sum_{e \in P_i} l_e^{i-1} (\epsilon \frac{\lambda^{i-1}}{c_e}) c_e \\ &= D^{i-1} + \epsilon \lambda^{i-1} \sum_{e \in P_i} l_e^{i-1} \\ &= D^{i-1} + \epsilon \lambda^{i-1} \alpha^{i-1} \qquad (1) \end{aligned}$$

where $\alpha^{i-1} = \sum_{e \in P_i} l_e^{i-1}$ can be seen as the length of the shortest path from $s$ to $t$ under length fn $l^{i-1}$.
Let $\beta$ be the minimum of all the value of dual objective functions we obtained. Therefore,

$$\begin{aligned} \beta &= \min_i \frac{D^i}{\alpha^i} \\ \implies \beta &\leq \frac{D^i}{\alpha^i} \quad \forall i \\ \implies \alpha^i &\leq \frac{D^i}{\beta} \quad \forall i \qquad (2) \end{aligned}$$

Therefore,

$$\begin{aligned} D^i &= D^{i-1} + \epsilon \lambda^{i-1} \alpha^{i-1} && \text{(from 1)} \\ &\leq D^{i-1} + \epsilon \lambda^{i-1} \frac{D^{i-1}}{\beta} && \text{(from 2)} \\ &= D^{i-1} (1 + \epsilon \frac{\lambda^{i-1}}{\beta}) \\ &\leq D^{i-1} e^{\epsilon \frac{\lambda^{i-1}}{\beta}} && \text{(Since } 1 + x \leq e^x) \end{aligned}$$

Now let us say that after $T$ iterations $D^T = 1$, then after $T$ iterations,

$$D^T \leq D^0 e^{\frac{\epsilon}{\beta} \sum_{i=1}^{T} \lambda^i} \qquad \text{(recursively)}$$

Observe that $\sum_{i=1}^{T} \lambda^i$ is the total flow we are sending from $s$ to $t$ in $T$ iterations because $\lambda^i$ is the flow we are sending in $i^{th}$ iteration. Let this be denoted by $F$. Therefore by replacing the sum by $F$ we get,

$$D^T \leq D^0 e^{\frac{\epsilon F}{\beta}}$$

Further by shifting terms and taking log both sides we get,

$$\begin{aligned} \frac{D^T}{D^0} &\leq e^{\frac{\epsilon F}{\beta}} \\ \implies \frac{\epsilon F}{\beta} &\geq \ln \frac{D^T}{D^0} \end{aligned}$$

Note that $D^0 = m\delta$ where $m$ is the number of edges and $D^T = 1$ from our stopping criterion. Therefore,

$$\frac{F}{\beta} \geq \frac{1}{\epsilon} \ln \frac{1}{m\delta} \tag{3}$$

**Claim:** There is a feasible flow of value $\frac{F}{\frac{1}{\epsilon} \ln \frac{1}{\delta}}$

*Proof:* Consider an edge $e$. For every $c_e$ units of flow through $e$ it's edge length is increased by a factor of at least $1 + \epsilon$. Let us assume that $f_e$ is the total flow sent through that edge. Therefore the edge length increases by a factor of at least $(1 + \epsilon)^{\frac{f_e}{c_e}}$

Note that $l_e^0 = \frac{\delta}{c_e}$ and $l_e^T \leq 1$ since $D^T = 1 = \sum_e l_e^T c_e$. Therefore,

$$(1 + \epsilon)^{\frac{f_e}{c_e}} \leq \frac{1}{\delta}$$

Further solving and taking log we get,

$$\frac{f_e}{c_e} \leq \frac{\ln \frac{1}{\delta}}{\ln (1 + \epsilon)}$$

From taylor series expansion we can say that $\ln (1 + \epsilon) \approx \epsilon$ for small values of $\epsilon$. Therefore we get,

$$\frac{f_e}{c_e} \leq \frac{1}{\epsilon} \ln \frac{1}{\delta}$$

Therefore congestion through each edge is at most $\frac{1}{\epsilon} \ln \frac{1}{\delta}$. Therefore scaling $F$ by this quantity gives us the feasible flow.

Value of flow therefore becomes $= \frac{F}{\frac{1}{\epsilon} \ln \frac{1}{\delta}}$

We want the flow to be close to the dual solution $\beta$. Therefore we want,

$$\begin{aligned} \frac{\beta}{F} \frac{1}{\epsilon} \ln \frac{1}{\delta} &\leq \frac{\frac{1}{\epsilon} \ln \frac{1}{\delta}}{\frac{1}{\epsilon} \ln \frac{1}{m\delta}} \leq 1 + \gamma \\ \implies \frac{1}{\delta} &\leq (\frac{1}{m\delta})^{1+\gamma} = \frac{1}{m\delta} (\frac{1}{m\delta})^{\gamma} \\ \implies (\delta)^{\gamma} &\leq \frac{1}{m^{1+\gamma}} \\ \implies \delta &\leq \frac{1}{m^{1+\frac{1}{\gamma}}} \end{aligned}$$

We can thus choose $\delta$ accordingly. Observe the trade-off. We cannot choose small $\delta$ as then it will take more iterations to converge to the solution and we can also not choose $\delta$ large as we also got an upper bound on $\delta$.