**COL758: Advanced Algorithms** | **Spring 2019**

# Lecture 4: January 14

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## 4.1 Learning With Expert Advice

Consider the scenario of horse race constituting two horses $\mathbf{H_1}, \mathbf{H_2}$ and m number of experts $\mathbf{E_1}, \mathbf{E_2}, ..., \mathbf{E_m}$ who gonna help us decide which horse to go with to win handsome amount of money in the bet. Each expert tells us the one horse who is going to win according to him/her in the particular race and then the final decision/bet is on us.
***AIM*** **:** Our aim is to bound the number of mistakes we do in long run of bettings.

### Case 1 : There is Always a Real Expert

In this case, we are guaranteed to have the perfect expert who can predict the outcome of every race correctly. Lets consider an algorithm known as ***Majority Algorithm***

#### Algorithm for Final Bet :-

1. Always go with what majority says
2. Remove any expert whose prediction was wrong in this round

**Note** : Since we are guaranteed of a real expert, we will never loose him in removal at any stage

#### Bound on Number of Mistakes

Consider the case when we commit mistake. At each instance, atleast half of the experts went wrong as we always go with the majority. And we are going to remove them, hence the number of experts will be atleast reduced by a factor of two. This process will continue till we are only left with the real expert.

$$\# \textbf{ mistakes} \leq \log_2 m$$

### Case 2: Every Expert Commits Mistakes

In the case 1, we had somehow unrealistic assumption that we always have a perfect expert. Here, we are relaxing this and now we have concept of the best expert i.e. an expert which commits minimum number of mistakes in the long run of betting. Lets consider an algorithm known as ***Weighted Majority Algorithm***

**Note:** The earlier mentioned algorithm based on removal of experts won't work in this case as even the best expert can go wrong, and we certainly don't want to loose him.

#### Algorithm for Final Bet :-

1. Assign each expert a weight $\mathbf{w_i}$, initialized to 1
2. Whenever $E_i$ commits a mistake, its weight is reduced by a factor of 2
3. Always go with the recommendation of the set of experts with larger weight

**Note** : Please observe that we are penalizing but not removing any of the expert

**Bound on Number of Mistakes**

- **OPT** = # mistakes done by the best expert in k rounds
- **W** = $\sum_1^m w_i$ (Note : Initial total weight = m)
- $\mathbf{W^i}$ = total weight at the end of $i^{th}$ step

Now consider two possible scenarios at each step :

1. If I make mistake

$$W^{i+1} \leq \frac{W^i}{2}(\frac{1}{2}) + \frac{W^i}{2} = 0.75W^i \text{ \textit{(Atleast half of the total weight corresponds to wrong choice)}}$$

2. If I don't commit mistake

$$W^{i+1} \leq W^i \text{ \textit{(Equal only when no expert commits mistake)}}$$

Suppose we ran for k steps and made t mistakes in total, then

$$W^k \leq (0.75)^t W^0$$
$$\text{Weight of the best expert} = \frac{1}{2^{OPT}}$$
$$W^k \geq \frac{1}{2^{OPT}} \text{ \textit{(as all weights are positive)}}$$

From the above equations, we get :-

$$\frac{1}{2^{OPT}} \leq (0.75)^t m$$

On simplifying, by taking $log_2$ on both sides, we get :-

$$t \leq \log_{\frac{4}{3}} m + \frac{OPT}{log_2 \frac{4}{3}}$$

## $\epsilon$ invariant of the above Algorithm

**Algorithm for Final Bet :-**

1. Assign each expert a weight $\mathbf{w_i}$ initialized to 1
2. Whenever $E_i$ commits a mistake, its weight is multiplied by a factor of $1 - \epsilon$ $(0 < \epsilon < 1)$
3. Always go with the recommendation of the set of experts with larger weight

**Bound on Number of Mistakes**

Consider two possible scenarios at each step :

1. If I make mistake

$$W^{i+1} \leq \frac{W^i}{2}(1-\epsilon) + \frac{W^i}{2} = W^i(1 - \frac{\epsilon}{2})$$

2. If I don't commit mistake

$$W^{i+1} \leq W^i$$

Suppose we ran for k steps and made t mistakes in total, then

$$W^k \leq (1-\frac{\epsilon}{2})^t W^0$$

$$\text{Weight of the best expert} = (1-\epsilon)^{OPT}$$

$$W^k \geq (1-\epsilon)^{OPT} \textit{ (as all weights are positive)}$$

From the above equations, we get :-

$$(1-\epsilon)^{OPT} \leq (1-\frac{\epsilon}{2})^t m$$

On simplifying, by taking $\log_{(1-\frac{\epsilon}{2})}$ on both sides, we get :-

$$OPT \log_{(1-\frac{\epsilon}{2})}(1-\epsilon) \geq t + \log_{(1-\frac{\epsilon}{2})} m$$

(Notice the sign of inequality changes as the base of logarithm is less than 1)

$$t \leq OPT \frac{\log_e(1-\epsilon)}{\log_e(1-\frac{\epsilon}{2})} - \frac{\log_e m}{\log_e(1-\frac{\epsilon}{2})}$$

We know the Taylor series expansion of $\log_e(1+x)$ as follows :

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + .....$$

Thus, if we choose our $\epsilon$ small enough, we can use the approximation $\log_e(1+x) = x$, hence the above expression simplifies as follows :-

$$t \leq 2(OPT) + 2\frac{\log_e m}{\epsilon}$$

## Randomized Weighted Majority Algorithm

In the above mentioned algorithm, we were making the final decision with certainty. Now, lets introduce the element of randomness. All algorithm remains same, just we will be deciding the final bet probabilistically i.e. if the weights of the two groups (for $H_1, H_2$ respectively) are 0.67 and 0.33, then we will be tossing a biased coin with probability distribution as follows :

$$\text{Probability(Head)} = 0.67$$
$$\text{Probability(Tails)} = 0.33$$

Now, if Head comes, I will bet on $H_1$, else I will bet on $H_2$. This is called betting by probability.
**NOTE :** Since we have introduced probability, we will be dealing with expectations. We will denote expectation of random variable X by **E**(X).

### Bound on Number of Mistakes

Let $X^i$ and $Y^i$ be the proportion of the $W^i$ corresponding to the wrong and right group respectively.

$$W^i = X^i + Y^i$$

$$\begin{aligned} W^{i+1} &= (1-\epsilon)X^i + Y^i \\ &= W^i(1-\epsilon\frac{X^i}{W^i}) \end{aligned}$$

Hence, we got

$$W^{i+1} = W^i(1-\epsilon(\text{Probabability of commiting mistake in } i^{th} \text{ step}))$$

Let's denote Probability of commiting mistake in $i^{th}$ step by $\mathbf{P}$(i). Hence, above equation can be rewritten as follows :

$$W^{i+1} = W^i(1 - \epsilon\mathbf{P}(i))$$

Now, we know the Taylor series expansion of $e^{-x}$ about $x = 0$ as follows :

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ......$$

Thus, above expression of $W^{i+1}$ can be transformed into following inequality :

$$\begin{aligned} W^{i+1} &\leq W^i e^{-\epsilon\mathbf{P}(i)} \\ &\leq W^{i-1}e^{-\epsilon(\mathbf{P}(i-1)+\mathbf{P}(i))} \\ &\leq W^0 e^{-\epsilon\sum_0^i \mathbf{P}(j)} \end{aligned}$$

Suppose we ran for k steps and made t mistakes in total, then

$$\sum_0^{k-1} \mathbf{P}(i) = \text{Expected number of mistakes in k steps} = \mathbf{E}(\#\text{ mistakes})$$

Thus, the above mentioned inequality can be rewritten as follows :-

$$W^k \leq me^{-\epsilon\mathbf{E}(\#\text{ mistakes})}$$

Also, the weight of the best expert $= (1-\epsilon)^{OPT}$, hence

$$W^k \geq (1-\epsilon)^{OPT}$$

Combining the above two equations, we get :

$$(1-\epsilon)^{OPT} \leq me^{-\epsilon\mathbf{E}(\#\text{ mistakes})}$$

On simplifying, by taking $\log_e$ on both sides, we get :-

$$OPT \log_e(1-\epsilon) \leq \log_e m - \epsilon\mathbf{E}(\#\text{ mistakes})$$

$$\mathbf{E}(\#\text{ mistakes}) \leq \frac{\log_e m}{\epsilon} - \frac{\log_e(1-\epsilon)}{\epsilon}OPT$$

We know the Taylor series expansion of $\log_e(1+x)$ as follows :

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + .....$$

Thus, if we choose our $\epsilon$ small enough, we can use the approximation $\log_e(1+x) = x - \frac{x^2}{2}$, hence the above expression simplifies as follows :-

$$\mathbf{E}(\#\text{ mistakes}) \leq OPT + \frac{\epsilon}{2}OPT + \frac{\log_e m}{\epsilon}$$

Observe the tradeoff we face for the choice of $\epsilon$. It is in numerator as well as denominator of the constituting terms of the final expression of the expected numnber of mistakes. We can choose the $\epsilon$ in the following way such that $\frac{\epsilon}{2}OPT$ and $\frac{\log_e m}{\epsilon}$ are almost equal.

$$\begin{aligned} \frac{\epsilon}{2}OPT &= \frac{\log_e m}{\epsilon} \\ \epsilon &= \sqrt{2\frac{\log_e m}{OPT}} \end{aligned}$$