Lecture 15: February 25

Lecturer: Naveen Garg *Scribe: Sukriti Gupta*

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15.1 Matching in Bipartite Graphs

We want to check if bipartite graph G=(U,V,E) has a perfect matching, where $|U|$ =$|V|$=n.

Consider the n × n matrix A, whose entries are given as

$$A(i,j) = \begin{cases} x_{ij}, & (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{matrix} & V \rightarrow \\ \begin{matrix} U \\ \downarrow \\ \\ \\ \end{matrix} & \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \end{matrix}$$

Det(A) is a multivariate polynomial (m variables) with degree n. Each term in the determinant (ignoring the sign) can be thought of as a permutation, with exactly 1 element of every row and 1 from every column. Each of these terms (monomials) has a degree n.

All permutations appear in the determinant (with some sign). This can be proved through induction. For a 2 × 2 matrix, only 2 permutations are possible. When we calculate the determinant of 3 × 3 matrices, we choose the first element of the permutation and take all possibilities, given that we chose that first element. This argument can be extended for all n. Let S_n be the set of all permutations.

$$Det(A) = \sum_{\sigma \in S_n} (-1)^{sign(\sigma)} \prod_{i=1}^{n} A(i, \sigma(i))$$

The determinant is **not** the zero polynomial **iff** $\exists$ a perfect matching in G.

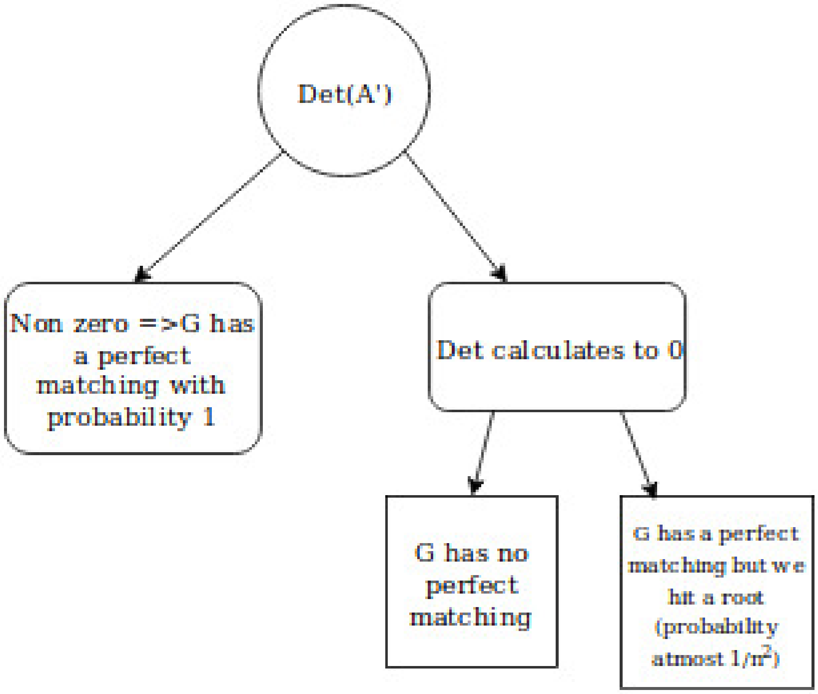
If there is a perfect matching, we can set all the variables corresponding to the edges in the matching to be 1 and all others 0. Then the determinant =1 or =-1 since there is only 1 non-zero term in the determinant. If the determinant is non-zero $\implies$ there is atleast one term which is not cancelled $\implies$ taking all the edges corresponding to this term (permutation) will make a perfect matching.

15.1.1 Schwartz-Zippel Lemma

Consider p($x_1, x_2, ..., x_n$) to be a non-zero degree d polynomial in n variables over a field $\mathbb{F}$. Let S be a subset of $\mathbb{F}$. Let x_i be picked randomly and independently from S.

$$P[p(x_1, x_2, ..., x_n) = 0] \leq \frac{d}{|S|}$$

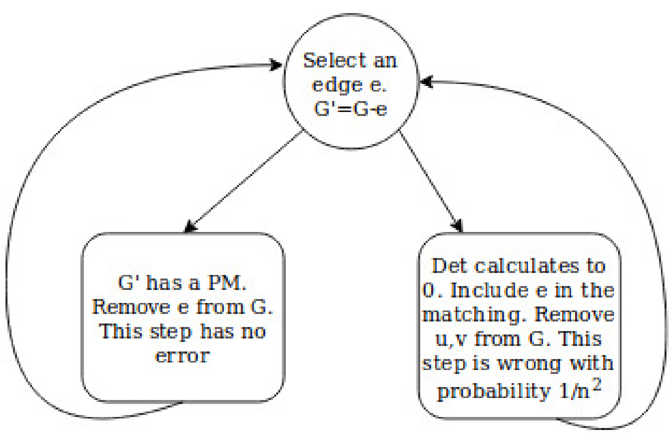
Take S to be the field $\mathbb{Z}$ mod p for some $p \geq n^3$. Let each x_{ij} be picked randomly and independently from S and its value substituted in A to give A$'$. Det(A) is a multivariate polynomial (m variables) with degree n.



Probability that G has a perfect matching but det(A′) = 0 ≤ $\frac{n}{n^3} = \frac{1}{n^2}$
Time required for computing determinant of an n × n matrix, inverting an n × n matrix, multiplying two n × n matrices is O(n^ω). The current best known algorithm for matrix multiplication has value of $\omega = 2.373$

15.1.2 How to find a perfect matching: Algorithm 1

Take an edge at random. Remove it. If determinant =0, the edge was in the matching (wrong with probability $\frac{1}{n^2}$), else, when determinant ≠ 0, the edge is not necessarily in the matching (always correct) (there is a perfect matching in G-e, so we can remove e).
This algorithm will take O(m n^ω) time.



Error: Since the matching has n edges, we go in the wrong conditon at most n times. The algorithm fails when there was a perfect matching but we could not find it.

$$P(Algorithm fails) \leq \frac{1}{n}$$

15.1.3 How to find a perfect matching: Algorithm 2

$$\begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & \dots & v_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \end{array}$$

We are trying to find v_j such that $(u_1, v_j) \in$ E and Det($A'_{-1,-j}$) $\neq 0$.

$$Det(A') = x_{11}Det(A'_{-1,-1}) \pm x_{12}Det(A'_{-1,-2}) \pm ... \pm x_{1n}Det(A'_{-1,-n})$$

If we have an A$'$ such that the det(A$'$) $\neq 0$, atleast one of these terms is non-zero. Take any of these non-zero terms, say, x_{1j} Det($A'_{-1,-j}$). Now, the edge represented by x_{1j} is in the perfect matching and we continue the process on $A'_{-1,-j}$. This can be done since Det($A'_{-1,-j}$) $\neq 0 \implies \exists$ a perfect matching in $A'_{-1,-j}$.

There is only one step in the algorithm where we can make a mistake, the first step that is finding an A$'$ such that det(A$'$) $\neq 0$.

$$P[error] = \frac{1}{n^2}$$

If we find that the determinant is still 0 even after k tries

$$P[error] = (\frac{1}{n^2})^k$$

To find the non-zero terms of det(A'), we can use Cramer's Rule, which states that the j^{th} element of the first column of $(A')^{-1}$ when multiplied to the det(A') gives the value of Det($A'_{-1,-j}$). We can therefore use the following algorithm:

matching $\leftarrow \phi$;
while *size of matching* $\neq n$ **do**
  If Det(A$'$)=0, exit;
  else B=$(A')^{-1}$
  find j such that $B_{j1} \neq 0$ and $A'_{1j} \neq 0$
  Include (1,j) in matching
  A$'$=$A'_{-1,-j}$
end

This algorithm will take O(n$^{\omega+1}$ + n^2) = O(n$^{\omega+1}$) time with success probability (probability that it found a perfect matching, given that it existed) = 1-$\frac{1}{n^2}$

15.2 Red Blue Matching Problem

Consider a bipartite graph G=(U,V,E) such that each edge e $\in$ E is either blue or red. Find a perfect matching with exactly k red edges. Does there exist a perfect matching with exactly k red edges?

Assumption: If there does exist a perfect matching with exactly k red edges, it is unique.

Consider the following matrix A.

$$A(i,j) = \begin{cases} y, & (i,j) \in \text{ E and is red} \\ 1, & (i,j) \in \text{ E and is blue} \\ 0, & \text{otherwise} \end{cases}$$

Det(A) is a polynomial in y of degree at most n. Coefficient of y^k is non-zero **iff** G has a red-blue perfect matching.

We can compute Det(A$'$) at n+1 points to find out the coefficients of Det(A).