COL758: Advanced Algorithms **Spring 2019**

Lecture 12: February 18

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12.1 Recap of Last Lecture

We saw the following $\widetilde{O}(m\sqrt{m})$ algorithm for the max-flow problem using electrical flows :

$y_i \leftarrow 1 \ \forall \ i$

repeat

Find x^r $s.t.$ $\sum_i y_i^r h_i^r(x^r) \leq 1$

$w^r \leftarrow \frac{1}{\max_i h_i(x^r)}$

$y_i \leftarrow y_i \cdot exp(\epsilon w^r h_i(x^r))$

until $\sum_r w^r \geq \frac{\ln m}{\epsilon^2}$;

Here, x^r is found by computing an electrical flow with resistances $x_i = y_i + \frac{\epsilon}{m}\sum_i y_i$. We also proved that No. of Rounds $\leq \frac{\rho \ln m}{\epsilon^2}$. Here ρ is the width of the algorithm and we showed $\rho \leq \sqrt{m}$ for the above procedure.

NOTE : $\widetilde{O}$ is the soft-O notation, in which we have ignored the ϵ and logarithmic terms.

12.2 Modification to a $\widetilde{O}(m^{4/3})$ Algorithm

We will now modify the above algorithm by trying to limit the width and consequently the #(Rounds) of the Algorithm.

Consider the following two modifications in our previous Algorithm :

1. If for some x^r , $x_i^r > (\frac{m \ln m}{\epsilon})^{1/3}$, then remove the edge i and recompute the electrical flow. The edge once removed will not be considered for any further rounds. The removal of an edge can also be interpreted as setting its resistance to infinity in the electrical flow.
Now , #(Electrical Flow Computations) = #(Rounds) + #(Edges Removed)

2. We add the following termination condition in our Algorithm : $y_i^r > m^{1/\epsilon}$, i.e. we stop the algorithm in round r, if the following condition is met for any i.

12.3 Analysis

First of all, let's see that in the end of the Algorithm we still have a valid flow for our network, i.e, the capacity constraints are satisfied :
Previously we proved ,

$$\sum_r w^r \geq \sum_r w^r \left(\sum_i y_i h_i(x^r) \right) \geq \max_i \sum_r w^r h_i(x^r) - \frac{\ln m}{\epsilon}$$

$$\Rightarrow \quad 1 \geq \max_i \frac{\sum_r w^r h_i(x^r)}{\sum_r w^r} - \frac{\ln m}{\epsilon \sum_r w^r} \qquad [A]$$

$$\text{Now , since} \quad h_i(\frac{\sum_r w^r x^r}{\sum_r w^r}) = \frac{\sum_r w^r h_i(x^r)}{\sum_r w^r} \quad \text{we have,}$$

$$\Rightarrow \quad 1 \geq \max_i h_i(\bar{x}) - \frac{\ln m}{\epsilon \sum_r w^r}$$

For ensuring that all capacity constraints are satisfied i.e. $h_i(\bar{x}) \leq 1 \; \forall i$, we want the last term in the above expression to be small , and therefore we had the termination condition : $\sum_r w^r \geq \frac{\ln m}{\epsilon^2}$.
Now for the new termination condition $y_i^r > m^{1/\epsilon}$,

$$y_i = \exp(\epsilon \sum_r w^r h_i(x^r)) > m^{\frac{1}{\epsilon}}$$

$$\Rightarrow \quad \epsilon \sum_r w^r h_i(x^r) > \frac{\ln m}{\epsilon}$$

$$\Rightarrow \quad \sum_r w^r h_i(x^r) > \frac{\ln m}{\epsilon^2}$$

$$\Rightarrow \quad \sum_r w^r h_i(x^r) > \frac{\ln m}{\epsilon^2}$$

$$\Rightarrow \quad \max_i \sum_r w^r h_i(x^r) > \frac{\ln m}{\epsilon^2}$$

Now, this with [A] gives ,

$$\max_i h_i(\bar{x}) \cdot (1-\epsilon) \leq 1 \quad \Rightarrow \quad \max_i h_i(\bar{x}) \leq 1+\epsilon$$

Therefore , we have shown that our algorithm still returns a flow which satisfies the capacity constraints.

Now, we bound the number of edge deletions, and in turn the number of electrical flow computations. For that , we will use the following three results :

1. $E(x^r) \quad \leq \quad (1+\epsilon)m^{1+\frac{1}{\epsilon}}$, where $E(x^r)$ is the energy of the flow x^r

2. $E(x^0) \quad \geq \quad \frac{1}{m^2}$, where $E(x^0)$ is the energy of the initial flow. For this we assume that our flow $F \geq 1$.

3. Every time an edge is deleted , the energy of the flow increases at least by a factor $(1 + \frac{\epsilon\bar{\rho}^2}{2m})$

Assuming the above, we achieve a bound on the number of edges deleted as follows :
Let H be the set of edges deleted, $h = |H|$

$$\left(1 + \frac{\epsilon\bar{\rho}^2}{2m}\right)^h \quad \leq \quad \frac{\text{Final Energy}}{\text{Initial Energy}} \quad \leq \quad \frac{(1+\epsilon)m^{1+\frac{1}{\epsilon}}}{\frac{1}{m^2}} \quad \approx \quad m^{3+\frac{1}{\epsilon}}$$

$$\Rightarrow \qquad h \quad \leq \quad \frac{(3+\frac{1}{\epsilon})\ln m}{\ln{(1+\frac{\epsilon\bar{\rho}^2}{2m})}}$$

Using $\ln{(1+x)} \quad \geq \quad \frac{x}{1+x} \quad :$

$$\Rightarrow \qquad h \quad \leq \quad \frac{(3+\frac{1}{\epsilon})\ln m}{\frac{\epsilon\bar{\rho}^2}{2m}} \cdot (1 + \frac{\epsilon\bar{\rho}^2}{2m})$$

Choosing ϵ such that $\quad 3 + \frac{1}{\epsilon} \quad \leq \quad \frac{2}{\epsilon} \quad$ we get,

$$\Rightarrow \qquad h \quad \leq \quad \frac{2m \cdot \frac{2}{\epsilon}\ln m}{\frac{1}{\epsilon^{2/3}} \cdot \epsilon \cdot m^{2/3} \cdot (\ln m)^{2/3}} \cdot \left(1 + \frac{\epsilon^{1/3}m^{2/3}(\ln m)^{2/3}}{2m}\right)$$

$$\Rightarrow \qquad h \quad \leq\approx \quad \frac{4(m\ln m)^{2/3}}{\epsilon^{4/3}} + \frac{2\ln m}{\epsilon}$$

In the above expression , the first term is the dominating term in the R.H.S. , so $h = \mathcal{O}\left(\frac{m^{1/3}(\ln m)^{1/3}}{\epsilon^{4/3}}\right)$

Also, $\quad \#(\text{Rounds}) \quad \leq \quad \frac{\bar{\rho}\ln m}{\epsilon^2} \quad = \quad \mathcal{O}\left(\frac{m^{1/3}(\ln m)^{4/3}}{\epsilon^{7/3}}\right) \quad$ because $\quad \rho \leq \left(\frac{m\ln m}{\epsilon}\right)^{1/3}$

Therefore overall ,

$\#(\text{Electrical Flow Computations}) = \#(\text{Rounds}) + \#(\text{Edges Removed}) = \mathcal{O}\left(\frac{m^{1/3}(\ln m)^{4/3}}{\epsilon^{7/3}}\right)$

$\Rightarrow$ The complete algorithm is $\mathcal{O}\left(\frac{m^{4/3}(\ln m)^{4/3}}{\epsilon^{7/3}}\right)$ i.e. $\widetilde{\mathcal{O}}(m^{4/3})$

Now we give a proof of the three results used :

1. $E(x^r) \quad \leq \quad (1+\epsilon)m^{1+\frac{1}{\epsilon}}$, where $E(x^r)$ is the energy of the flow x^r.

Let x^* be the flow in $\mathcal{Q}$ which satisfies the capacity constraints. Then,

$$E(x) \quad \leq \quad E(x^*) \quad = \quad \sum_i f_i^2(x^*) \cdot R_i \quad \leq \quad \sum_i R_i \quad = \quad (1+\epsilon)\sum_i y_i$$

Now, our termination condition ensures that $y_i \leq m^{1/\epsilon}\ \forall i$. This gives the required result since $\sum_i y_i \leq m^{1+1/\epsilon}$

2. $E(x^0) \quad \geq \quad \frac{1}{m^2}$, where $E(x^0)$ is the energy of the initial flow.

This is trivial because since there are m edges , and the total flow is assumed to be greater than m, there is at least one edge with flow at least $1/m$. Also initially all the resistances are 1. So this edge contributes $1/m^2$ to the energy and the result follows.

3. Every time an edge is deleted , the energy of the flow increases at least by a factor $(1 + \frac{\epsilon\bar{\rho}^2}{2m})$

Let x be an electrical flow after which an edge e was deleted and x' be the new flow computed just after the deletion. Let E, E' be the energies of x, x' respectively.

Claim : $E' \quad \geq \quad E + f_e^2(x)r_e(x)$

Assuming the above claim to be true, we have :

Since $f_e(x) \geq \bar{\rho}$, $E' \geq E + \frac{\bar{\rho}^2\epsilon\sum_i y_i}{m}$

$\Rightarrow \quad E' \quad \geq \quad E + \frac{\bar{\rho}^2\epsilon E}{m(1+\epsilon)} \quad \geq \quad E\left(1 + \frac{\bar{\rho}^2\epsilon}{2m}\right)$

The proof for the claim used above is left as a homework.