The Linear Complementarity Problem and Lemke's Algorithm

November 11, 2014

Given an $n \times n$ matrix $\boldsymbol{M}$, and a vector $\boldsymbol{q}$, the linear complementarity problem[1] asks for a vector $\boldsymbol{y}$ satisfying the following conditions:

$$\boldsymbol{My} \leq \boldsymbol{q}, \quad \boldsymbol{y} \geq 0 \quad \text{and} \quad \boldsymbol{y} \cdot (\boldsymbol{q} - \boldsymbol{My}) = 0. \tag{1}$$

The problem is interesting only when $\boldsymbol{q} \not\geq 0$, since otherwise $\boldsymbol{y} = 0$ is a trivial solution. Let us introduce slack variables $\boldsymbol{v}$ to obtain the equivalent formulation.

$$\boldsymbol{My} + \boldsymbol{v} = \boldsymbol{q}, \quad \boldsymbol{y} \geq 0, \quad \boldsymbol{v} \geq 0 \quad \text{and} \quad \boldsymbol{y} \cdot \boldsymbol{v} = 0. \tag{2}$$

The reason for imposing non-negativity on the slack variables is that the first condition in (1) implies $\boldsymbol{q} - \boldsymbol{My} \geq 0$. Let $\mathcal{P}$ be the polyhedron in $2n$ dimensional space defined by the first three conditions; we will assume that $\mathcal{P}$ is non-degenerate[2]. Under this condition, any solution to (2) will be a vertex of $\mathcal{P}$, since it must satisfy $2n$ equalities. Note that the set of solutions may be disconnected.

An ingenious idea of Lemke was to introduce a new variable and consider the system, which is called the *augmented LCP*:

$$\boldsymbol{My} + \boldsymbol{v} - z\boldsymbol{1} = \boldsymbol{q}, \quad \boldsymbol{y} \geq 0, \quad \boldsymbol{v} \geq 0, \quad z \geq 0 \quad \text{and} \quad \boldsymbol{y} \cdot \boldsymbol{v} = 0. \tag{3}$$

Let $\mathcal{P}'$ be the polyhedron in $2n + 1$ dimensional space defined by the first four conditions of the augmented LCP; again we will assume that $\mathcal{P}'$ is non-degenerate. Since any solution to (3) must still satisfy $2n$ equalities, the set of solutions, say S, will be a subset of the one-skeleton of $\mathcal{P}'$, i.e., it will consist of edges and vertices of $\mathcal{P}'$. Any solution to the original system must satisfy the additional condition $z = 0$ and hence will be a vertex of $\mathcal{P}'$.

Now S turns out to have some nice properties. Any point of S is *fully labeled* in the sense that for each i, $y_i = 0$ or $v_i = 0$.[3] We will say that a point of S *has double label* i if $y_i = 0$ and $v_i = 0$ are both satisfied at this point. Clearly, such a point will be a vertex of $\mathcal{P}'$ and it will have only one double label. Since there are exactly two ways of relaxing this double label, this vertex must have exactly two edges of S incident at it. Clearly, a solution to the original system (i.e., satisfying $z = 0$) will be a vertex of $\mathcal{P}'$ that does not have a double label. On relaxing $z = 0$, we get the unique edge of S incident at this vertex.

As a result of these observations, it follows that S consists of paths and cycles. Of these paths, Lemke's algorithm explores a special one. An unbounded edge of S such that the vertex of $\mathcal{P}'$ it is incident on has $z > 0$ is called a *ray*. Among the rays, one is special – the one on which $\boldsymbol{y} = 0$. This is called the *primary ray* and the rest are called *secondary rays*. Now Lemke's algorithm explores, via pivoting, the path starting with the primary ray. This path must end either in a vertex satisfying $z = 0$, i.e., a solution to the original system, or a secondary ray. In the latter case, the algorithm is unsuccessful in finding a solution to the original system; in particular, the original system may not have a solution.

Remark: Observe that $z\boldsymbol{1}$ can be replaced by $z\boldsymbol{a}$, where vector $\boldsymbol{a}$ has a 1 in each row in which $\boldsymbol{q}$ is negative and has either a 0 or a 1 in the remaining rows, without changing its role; in our algorithm,

[1] We refer the reader to [1] for a comprehensive treatment of notions presented in this section.

[2] A polyhedron in n-dimension is said to be *non-degenerate* if on its d-dimensional face exactly $n - d$ of its constraints hold with equality. For example on vertices (0-dimensional face) exactly n constraints hold with equality. There are many other equivalent ways to describe this notion.

[3] These are also known as *almost complementary solutions* in the literature.

we will set a row of $\boldsymbol{a}$ to 1 if and only if the corresponding row of $\boldsymbol{q}$ is negative. As mentioned above, if $\boldsymbol{q}$ has no negative components, (1) has the trivial solution $\boldsymbol{y} = 0$. Additionally, in this case Lemke's algorithm cannot be used for finding a non-trivial solution, since it is simply not applicable. However, Lemke-Howson scheme is applicable for such a case; it follows a complementary path in the original polyhedron (2) starting at $\boldsymbol{y} = 0$, and guarantees termination at a non-trivial solution if the polyhedron is bounded.

References

[1] Cottle, R., Pang, J., Stone, R.: The Linear Complementarity Problem. Academic Press, Boston (1992)