Uncertainty

Chapter 13

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(Based on slides by UW-AI faculty)

Logic vs. Probability

Symbol: Q, R …	**Random variable:** Q …
Boolean values: T, F	**Domain:** you specify e.g. {heads, tails} [1, 6]
State of the world: Assignment to Q, R … Z	**Atomic event:** complete specification of world: Q… Z • Mutually exclusive • Exhaustive
	Prior probability (aka Unconditional prob: P(Q)
	Joint distribution: Prob. of every atomic event

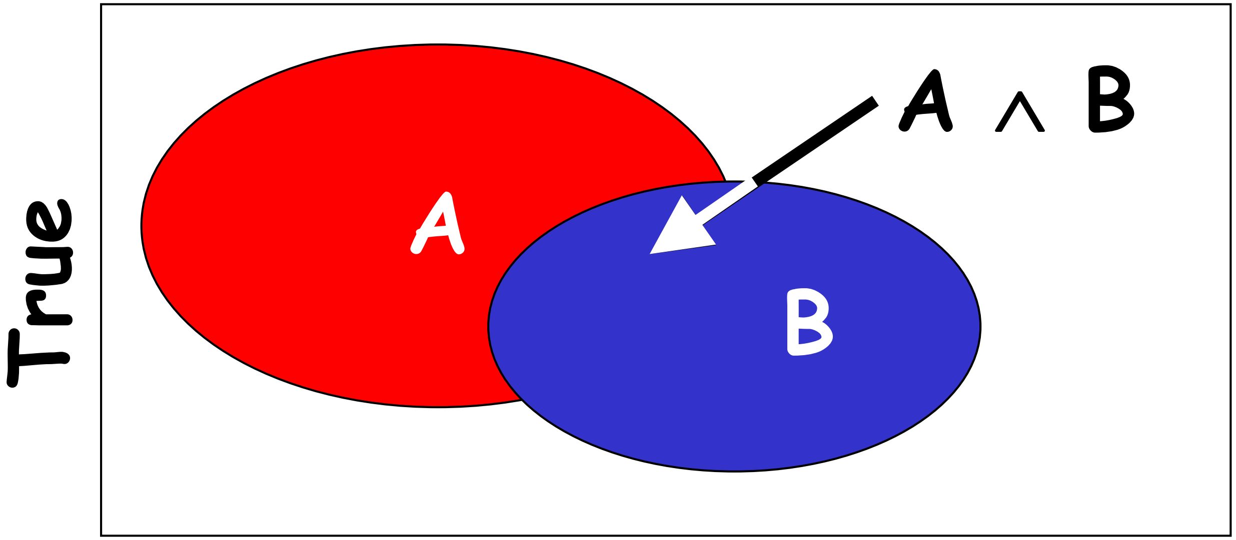
Probability Basics

- Begin with a set S: the sample space
  - e.g., 6 possible rolls of a die.
- x ϵ S is a sample point/possible world/atomic event
- A probability space or probability model is a sample space with an assignment P(x) for every x s.t. $0 \leq P(x) \leq 1$ and $\sum P(x) = 1$
- An event A is any subset of S
  - e.g. A= ‘die roll < 4’
- A random variable is a function from sample points to some range, e.g., the reals or Booleans

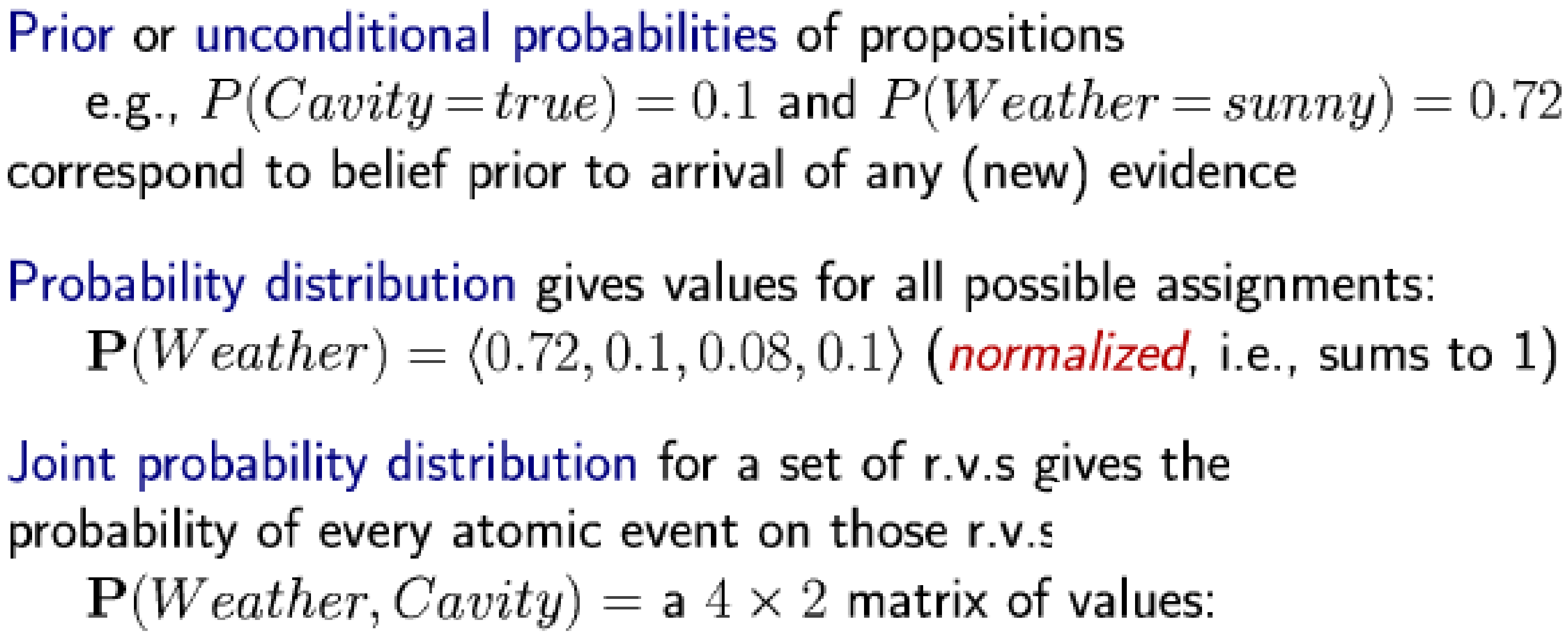
Axioms of Probability Theory

- All probabilities between 0 and 1
  - $0 \leq P(A) \leq 1$
  - P(true) = 1
  - P(false) = 0.
- The probability of disjunction is:

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$



Prior Probability



Prior or unconditional probabilities of propositions

e.g., $P(Cavity = true) = 0.1$ and $P(Weather = sunny) = 0.72$

correspond to belief prior to arrival of any (new) evidence

Probability distribution gives values for all possible assignments:

$\mathbf{P}(Weather) = \langle 0.72, 0.1, 0.08, 0.1 \rangle$ (*normalized*, i.e., sums to 1)

Joint probability distribution for a set of r.v.s gives the probability of every atomic event on those r.v.s

$\mathbf{P}(Weather, Cavity) =$ a 4×2 matrix of values:

Joint distribution can answer any question

Conditional probability

- Conditional or posterior probabilities

  e.g., P(*cavity* | *toothache*) = 0.8

  i.e., given that *toothache* is all I know there is 80% chance of cavity

- Notation for conditional distributions:

  P(*Cavity* | *Toothache*) = 2-element vector of 2-element vectors)

- If we know more, e.g., *cavity* is also given, then we have

  P(*cavity* | *toothache, cavity*) = 1

- New evidence may be irrelevant, allowing simplification:

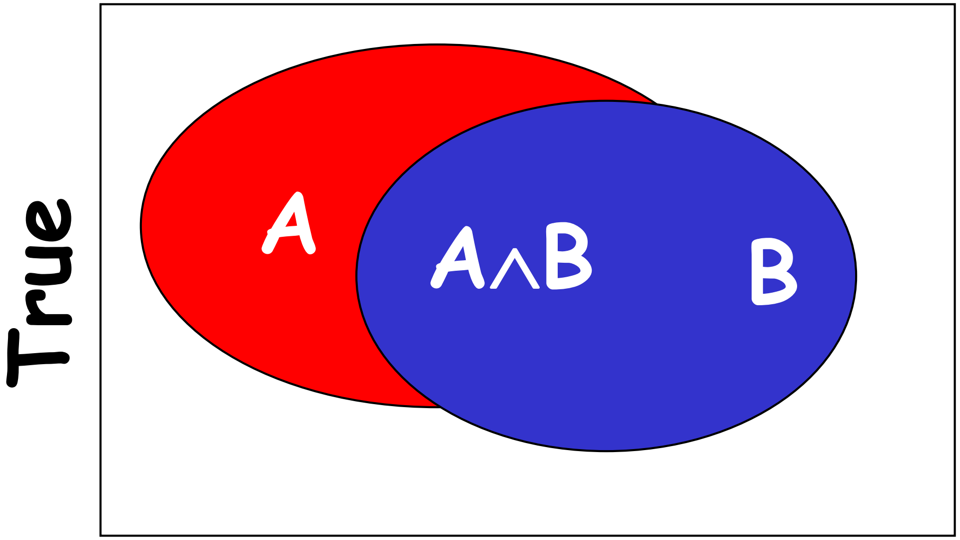
  P(*cavity* | *toothache, sunny*) = P(*cavity* | *toothache*) = 0.8

- This kind of inference, sanctioned by domain knowledge, is crucial

Conditional Probability

- P(*A* | *B*) is the probability of *A* given *B*
- Assumes that *B* is the only info known.
- Defined by:

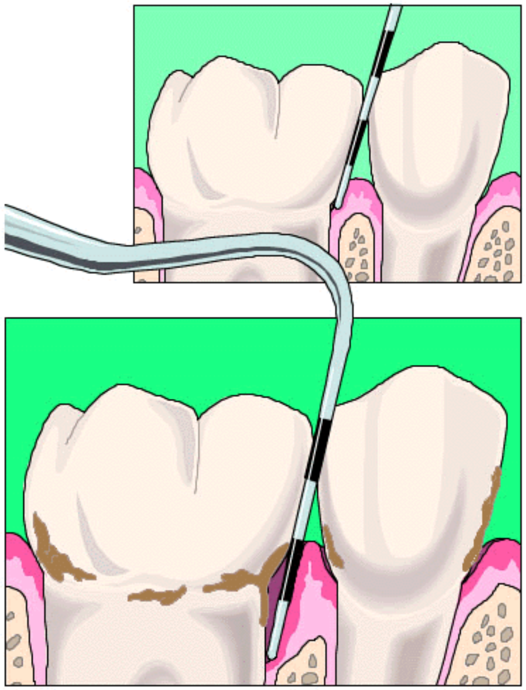
$$P(A \mid B) = \frac{P(A \wedge B)}{P(B)}$$



Chain Rule/Product Rule

- $P(X_1, \ldots, X_n) = P(X_n | X_1..X_{n-1}) P(X_{n-1} | X_1..X_{n-2}) \ldots P(X_1)$
  $= \Pi P(X_i | X_1,..X_{i-1})$

Dilemma at the Dentist’s



What is the probability of a cavity given a toothache?
What is the probability of a cavity given the probe catches?

Inference by Enumeration

Start with the joint distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	[redacted]	[redacted]	.072	.008
¬ *cavity*	[redacted]	[redacted]	.144	.576

For any proposition ϕ, sum the atomic events where it is true:

$$P(\phi) = \Sigma_{\omega:\omega\models\phi} P(\omega)$$

P(toothache)=.108+.012+.016+.064

= .20 or 20%

Inference by Enumeration

Start with the joint distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	[redacted]	[redacted]	[redacted]	[redacted]
¬ *cavity*	[redacted]	[redacted]	**.144**	**.576**

For any proposition ϕ, sum the atomic events where it is true:

$$P(\phi) = \Sigma_{\omega:\omega\models\phi} P(\omega)$$

$P(toothache \vee cavity) = .20 + .072 + .008$

.28

Inference by Enumeration

Start with the joint distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	**.108**	**.012**	**.072**	**.008**
¬ cavity	**.016**	**.064**	**.144**	**.576**

Can also compute conditional probabilities:

$$
\begin{aligned}
P(\neg cavity|toothache) &= \frac{P(\neg cavity \wedge toothache)}{P(toothache)} \\
&= \frac{0.016+0.064}{0.108+0.012+0.016+0.064} = 0.4
\end{aligned}
$$

Complexity of Enumeration

- Worst case time: $O(d^n)$
  - Where d = max arity
  - And n = number of random variables
- Space complexity also $O(d^n)$
  - Size of joint distribution

- Prohibitive!

Independence

- *A* and *B* are *independent* iff:

$$P(A \mid B) = P(A)$$

$$P(B \mid A) = P(B)$$

These two constraints are logically equivalent

- Therefore, if *A* and *B* are independent:

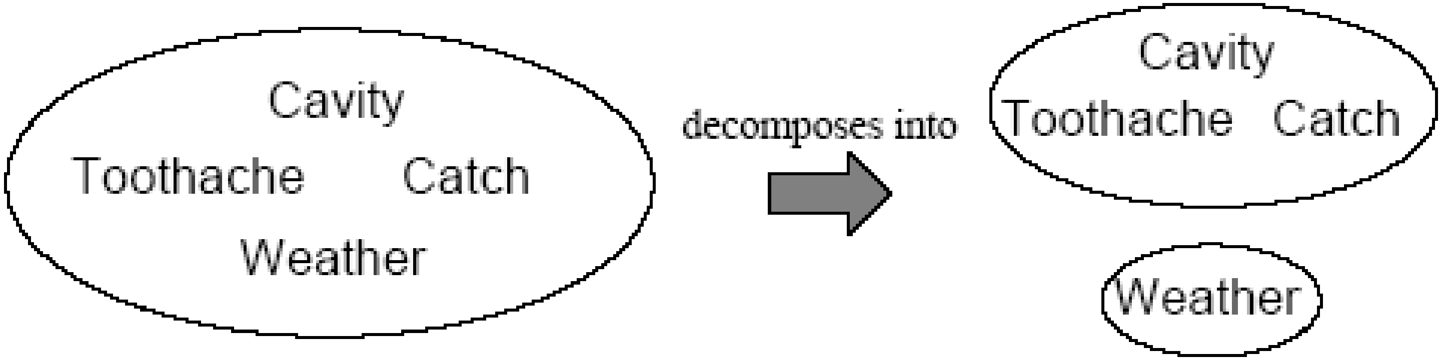
$$P(A \mid B) = \frac{P(A \wedge B)}{P(B)} = P(A)$$

$$P(A \wedge B) = P(A)P(B)$$

Independence

A and B are independent iff

$\mathbf{P}(A|B) = \mathbf{P}(A)$ or $\mathbf{P}(B|A) = \mathbf{P}(B)$ or $\mathbf{P}(A, B) = \mathbf{P}(A)\mathbf{P}(B)$



$$\mathbf{P}(Toothache, Catch, Cavity, Weather)$$
$$= \mathbf{P}(Toothache, Catch, Cavity)\mathbf{P}(Weather)$$

32 entries reduced to 12; for n independent biased coins, $2^n \rightarrow n$

Complete independence is powerful but rare

What to do if it doesn't hold?

Conditional Independence

$\mathbf{P}(Toothache, Cavity, Catch)$ has $2^3 - 1 = 7$ independent entries

If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

(1) $P(catch|toothache, cavity) = P(catch|cavity)$

The same independence holds if I haven't got a cavity:

(2) $P(catch|toothache, \neg cavity) = P(catch|\neg cavity)$

$Catch$ is *conditionally independent* of $Toothache$ given $Cavity$:

$$\mathbf{P}(Catch|Toothache, Cavity) = \mathbf{P}(Catch|Cavity)$$

Instead of 7 entries, only need 5

Conditional Independence II

P(catch | toothache, cavity) = P(catch | cavity)

P(catch | toothache,¬cavity) = P(catch |¬cavity)

Equivalent statements:

$$\mathbf{P}(Toothache|Catch, Cavity) = \mathbf{P}(Toothache|Cavity)$$
$$\mathbf{P}(Toothache, Catch|Cavity) = \mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)$$

Why only 5 entries in table?

Write out full joint distribution using chain rule:

$$\begin{aligned}&\mathbf{P}(Toothache, Catch, Cavity)\\ &= \mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch, Cavity)\\ &= \mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)\\ &= \mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)\end{aligned}$$

I.e., $2 + 2 + 1 = 5$ independent numbers (equations 1 and 2 remove 2)

Power of Cond. Independence

- Often, using conditional independence reduces the storage complexity of the joint distribution from exponential to linear!!

- Conditional independence is the most basic & robust form of knowledge about uncertain environments.

Bayes Rule

Bayes rules!

posterior

$$P(x,y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

Useful for assessing diagnostic probability from causal probability:

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

Computing Diagnostic Prob. from Causal Prob.

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

E.g. let M be meningitis, S be stiff neck

P(M) = 0.0001,

P(S) = 0.1,

P(S|M)= 0.8

P(M|S) $= \frac{P(s|m)P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008$

Note: posterior probability of meningitis still very small!

Other forms of Bayes Rule

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{\sum_x P(y \mid x)\ P(x)}$$

$$P(x \mid y) = \alpha P(y \mid x) P(x)$$

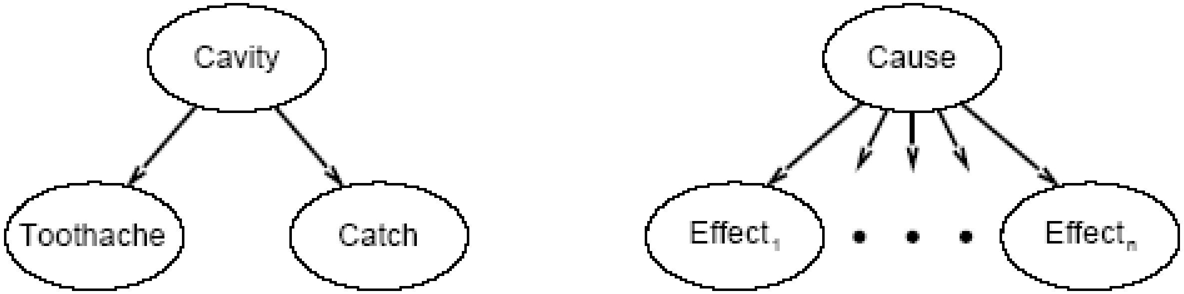
$$\text{posterior} \propto \text{likelihood} \cdot \text{prior}$$

Bayes’ Rule & Cond. Independence

$$
\begin{aligned}
&\mathbf{P}(Cavity|toothache \wedge catch)\\
&= \alpha\,\mathbf{P}(toothache \wedge catch|Cavity)\mathbf{P}(Cavity)\\
&= \alpha\,\mathbf{P}(toothache|Cavity)\mathbf{P}(catch|Cavity)\mathbf{P}(Cavity)
\end{aligned}
$$

This is an example of a *naive Bayes* model:

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause)\Pi_i\mathbf{P}(Effect_i|Cause)$$



Total number of parameters is *linear* in n