2-4 Trees

COL 106

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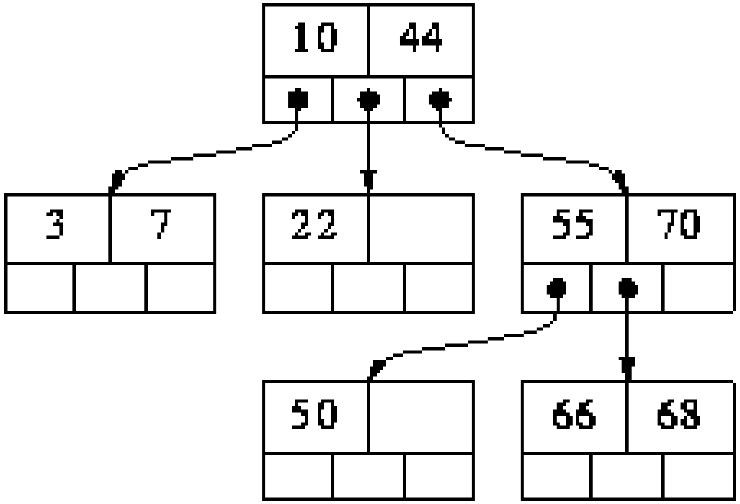
Multi-Way Trees

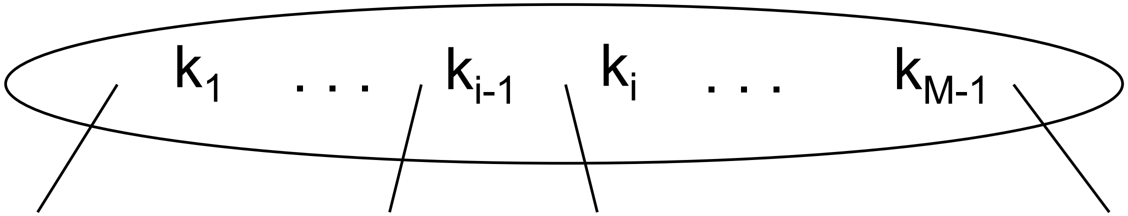


- A binary search tree:
  - *One* value in each node
  - At most 2 children
- An *M-way* search tree:
  - Between *1* to *(M-1)* values in each node
  - At most *M* children per node

M-way Search Tree Details

Each internal node of an *M-way* search has:

- Between *1* and *M* children
- Up to *M-1* keys $k_1 , k_2 , \ldots , k_{M-1}$

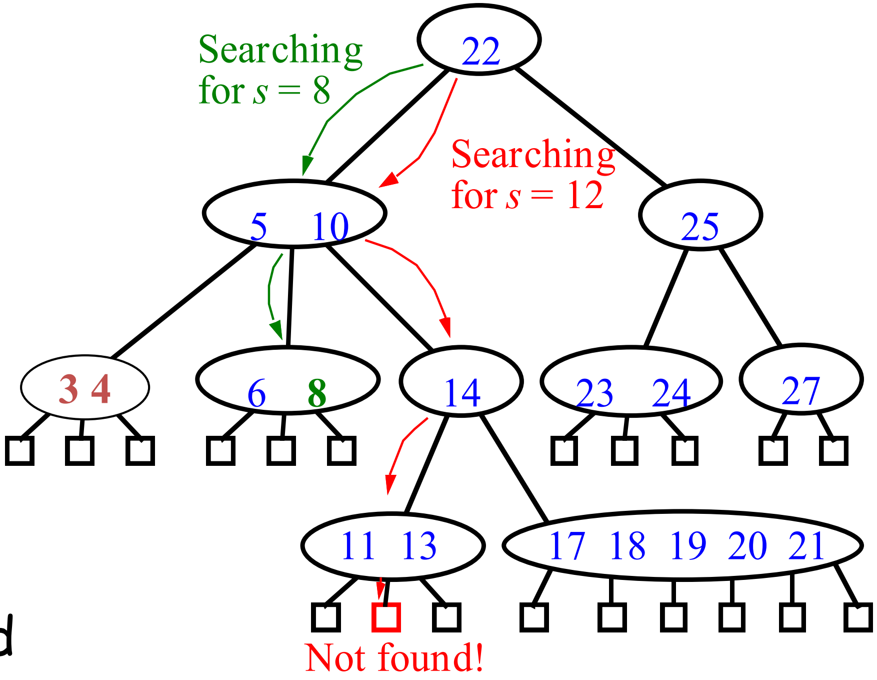


Keys are ordered such that:

$k_1 < k_2 < \ldots < k_{M-1}$

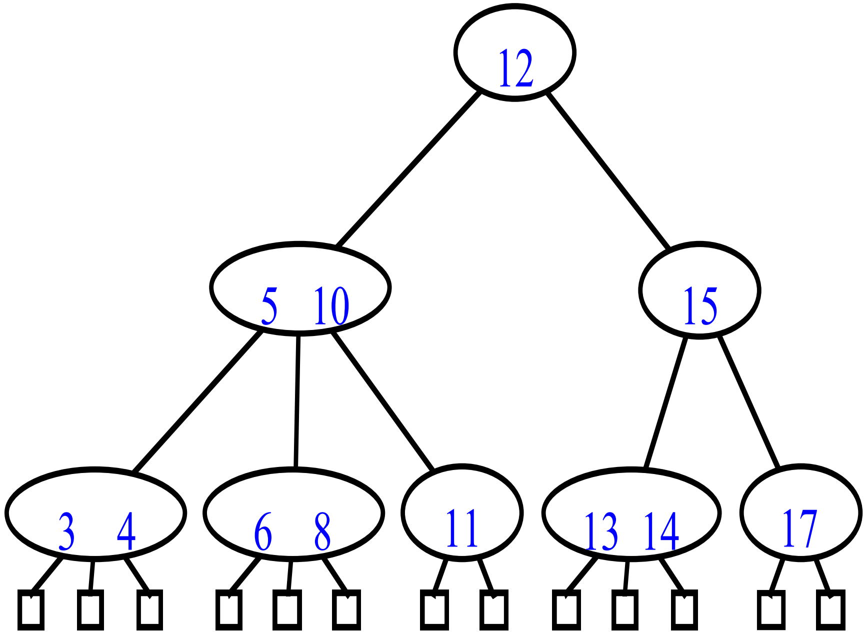
Multi-way Searching

- Similar to binary searching
  - If search key $s<k_1$ search the leftmost child
  - If $s>k_{d-1}$, search the rightmost child
- Multiway search tree ?
  - Find two keys k_{i-1} and k_i between which s falls, and search the child v_i.
- What would an in-order traversal look like?



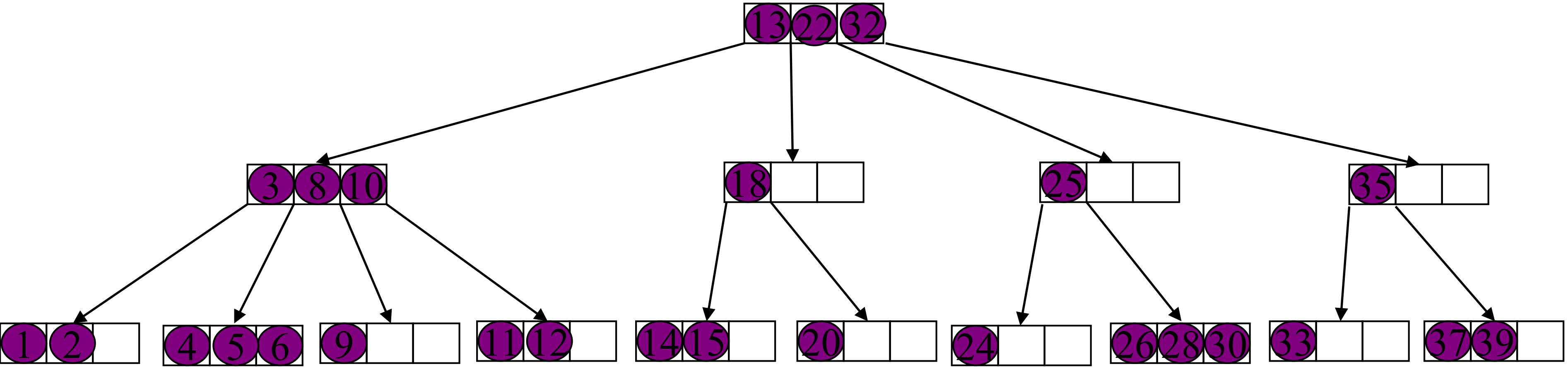
(2,4) Trees

- Properties:
  - Each node has at most 4 children
  - All external nodes have same depth
  - Height *h* of (2,4) tree is *O*(log *n*).
- How is the last fact useful in searching?

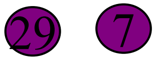


Insertion

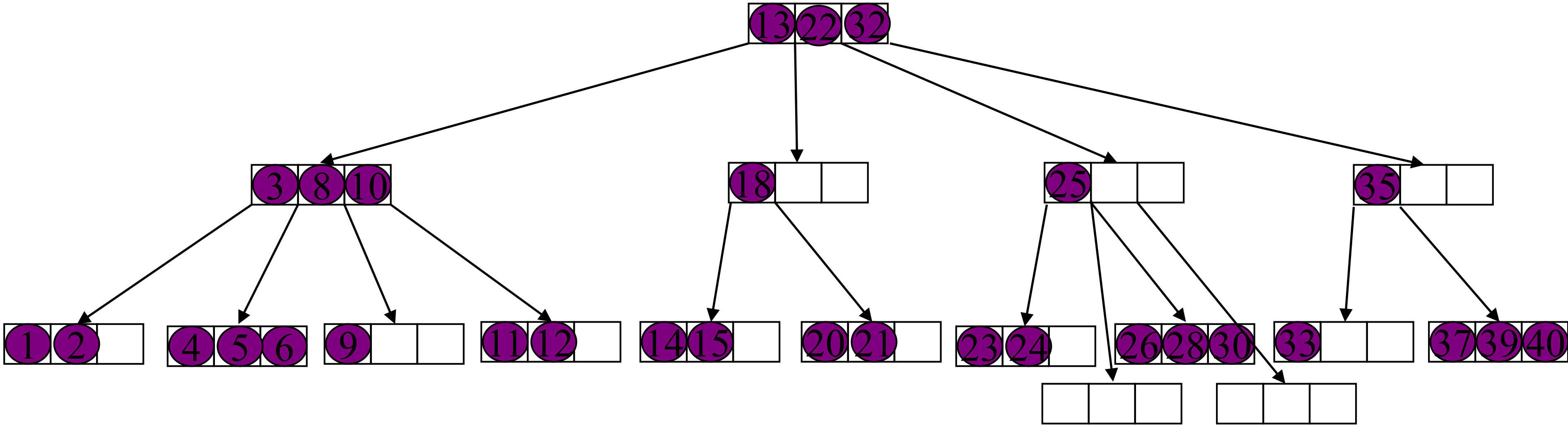
- No problem if the node has empty space



Insertion(2)

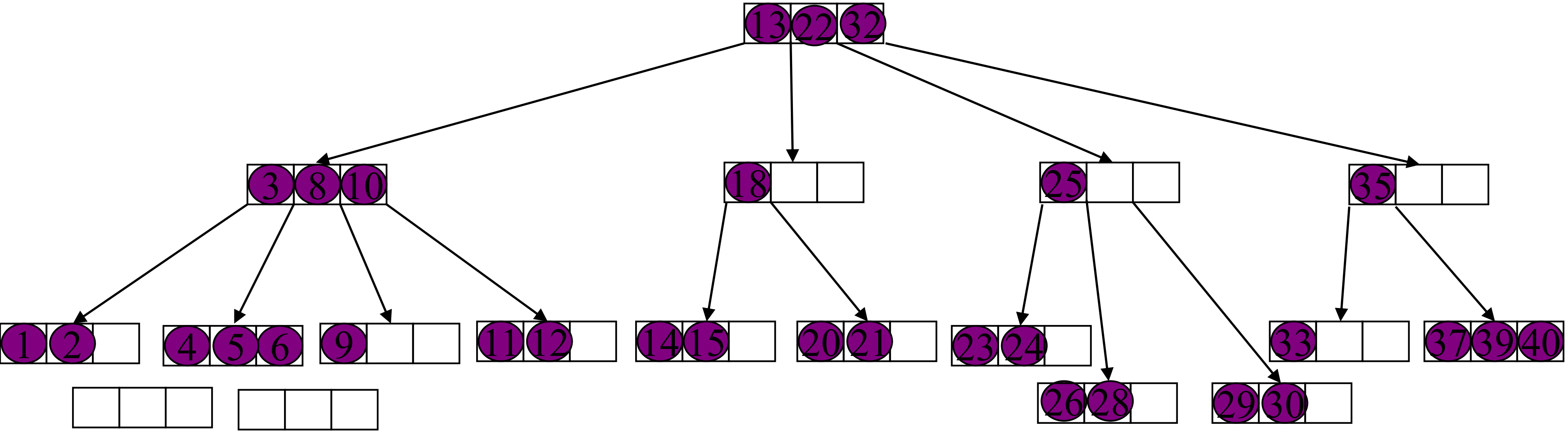


- Nodes get split if there is insufficient space.



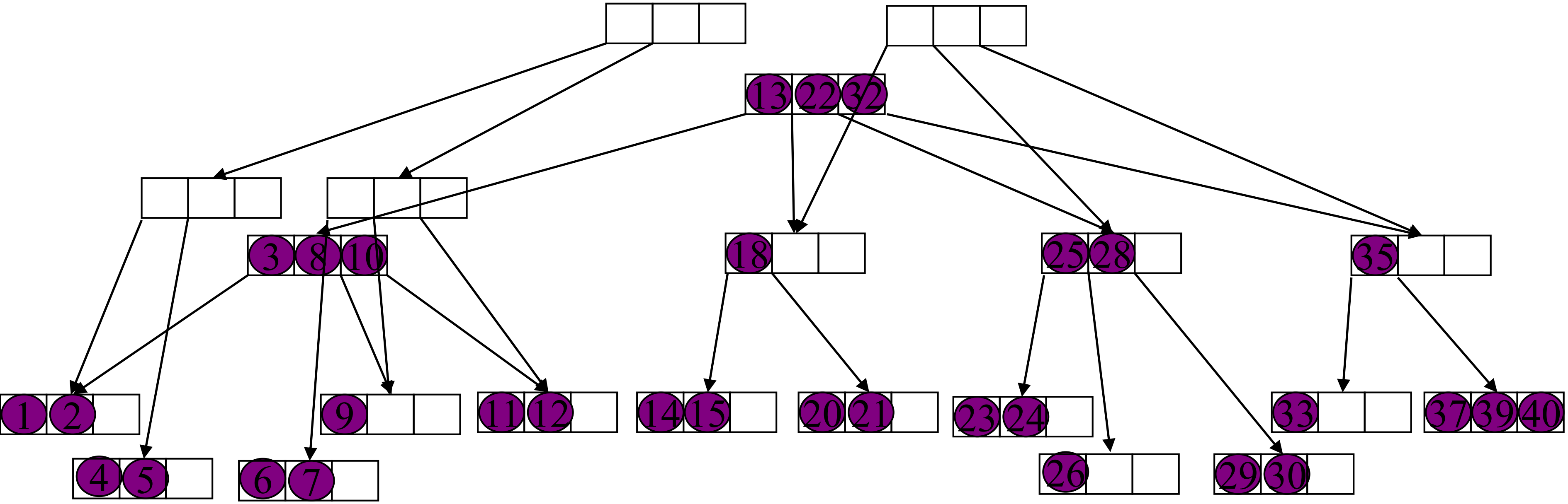
Insertion(3)

- One key is promoted to parent and inserted in there



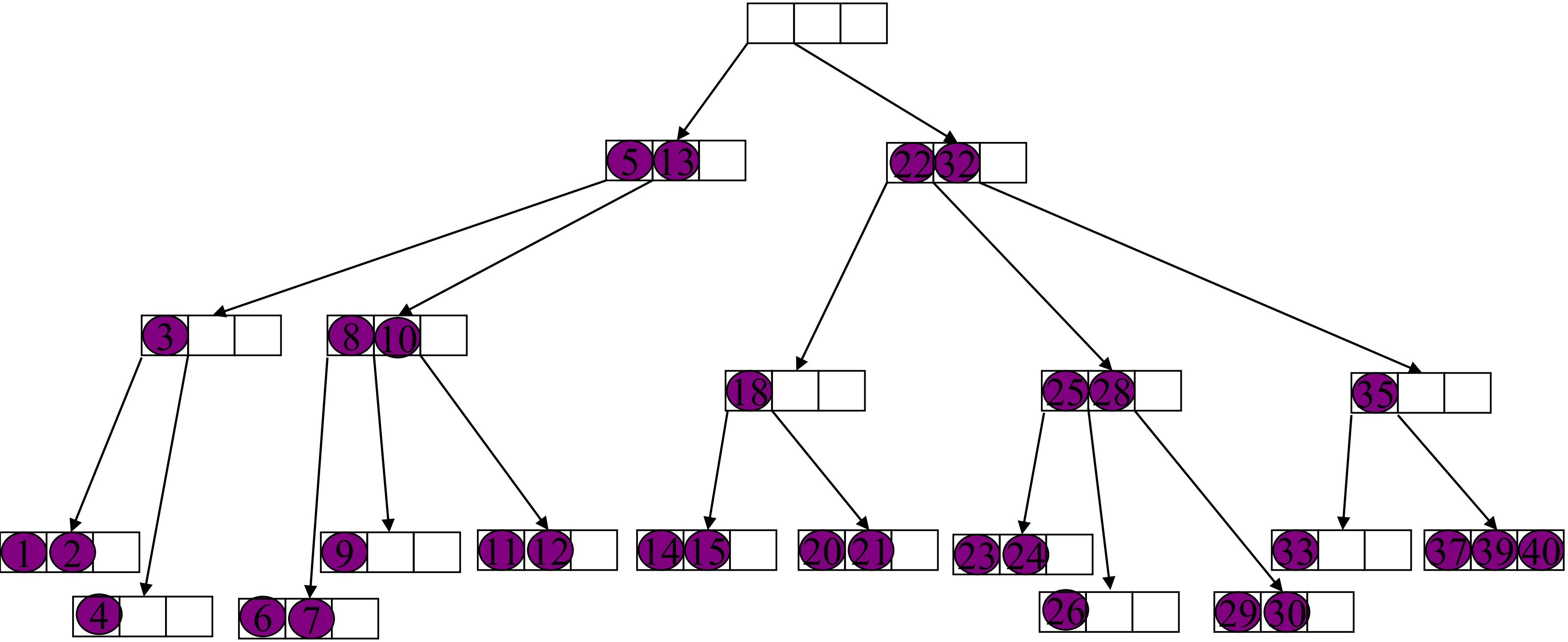
Insertion(4)

- If parent node does not have sufficient space then it is split.
- In this manner splits can cascade.



Insertion(5)

- Eventually we may have to create a new root.
- This increases the height of the tree

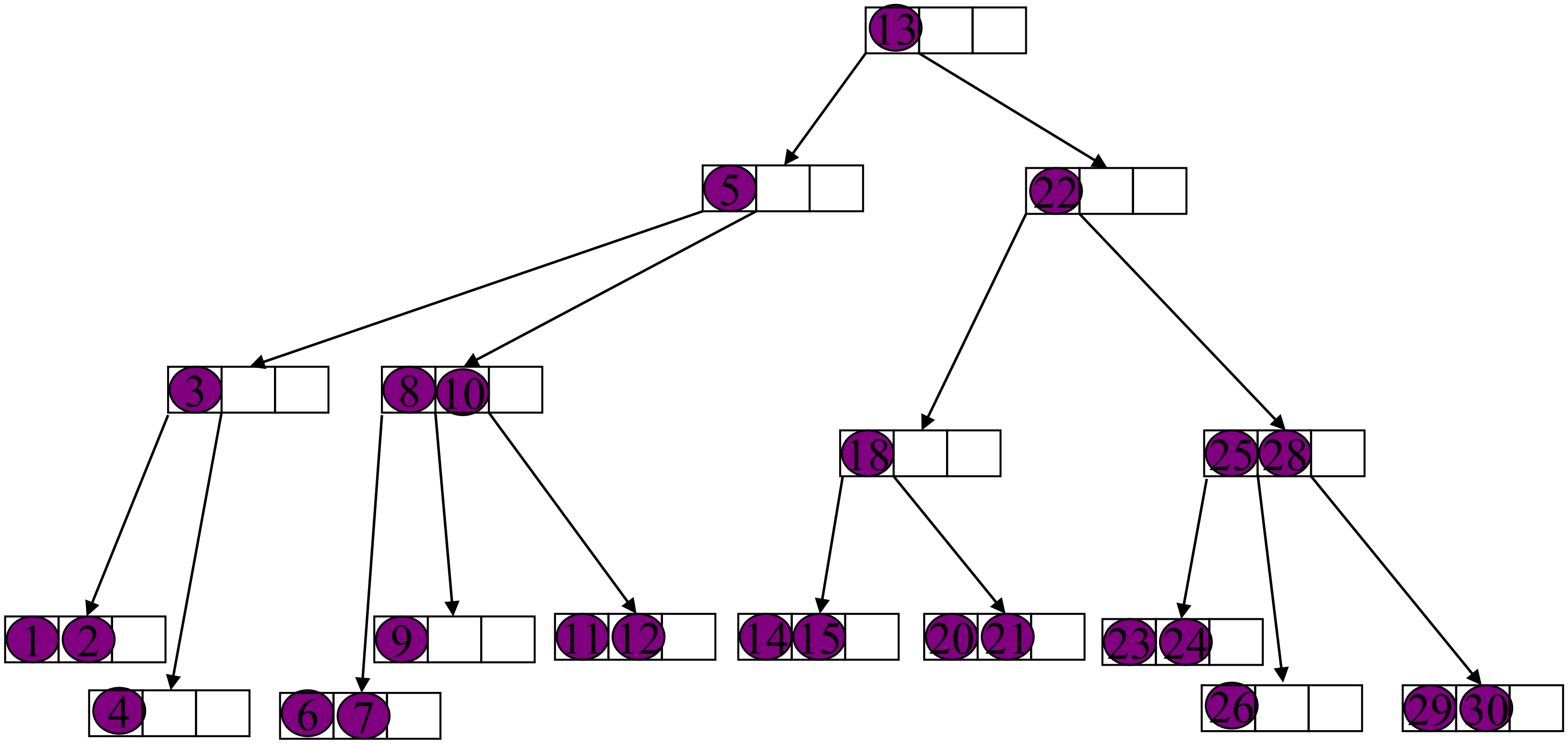


Time for Search and Insertion

- A search visits O(log N) nodes
- An insertion requires O(log N) node splits
- Each node split takes constant time
- Hence, operations Search and Insert each take time O(log N)

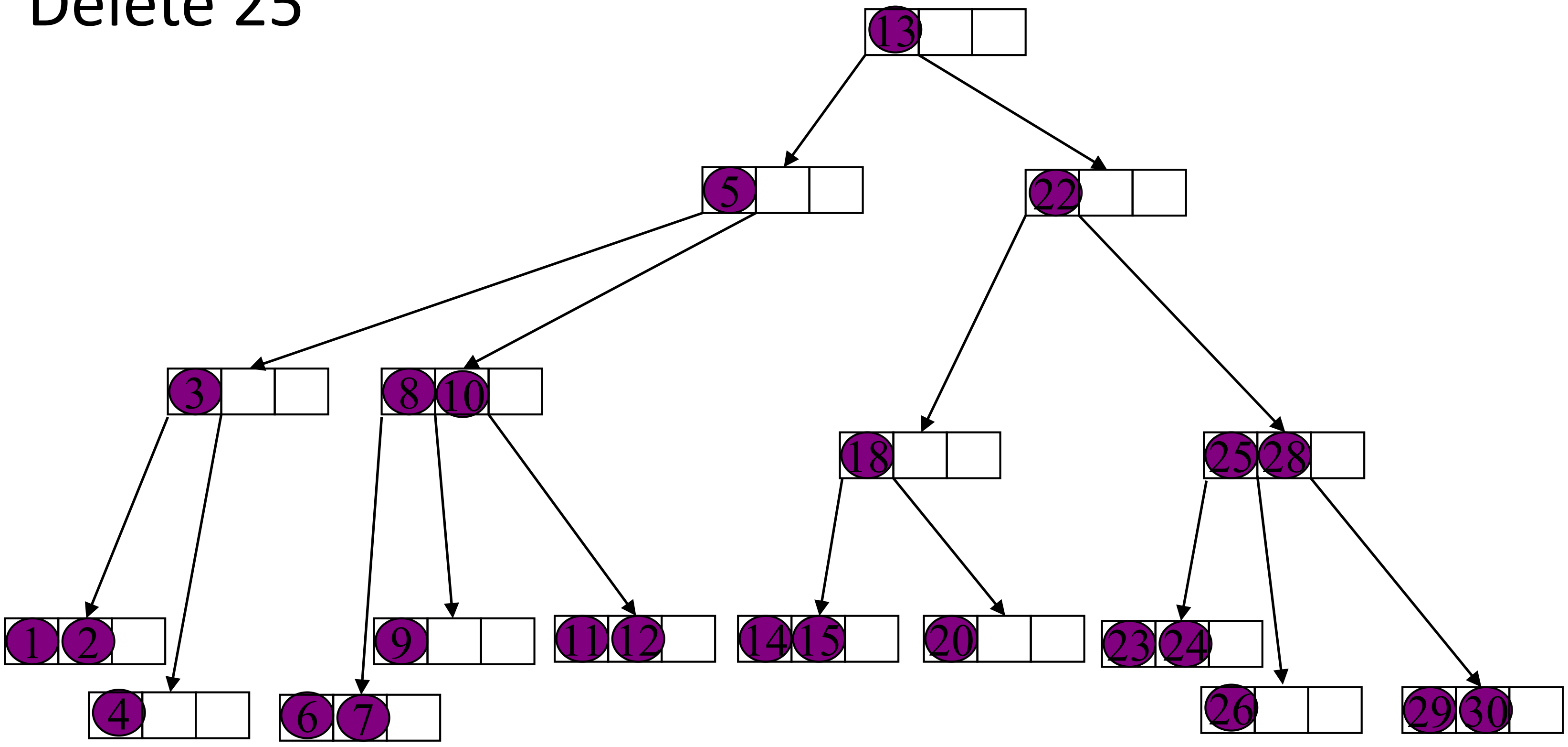
Deletion

- Delete 21.
- No problem if key to be deleted is in a leaf with at least 2 keys



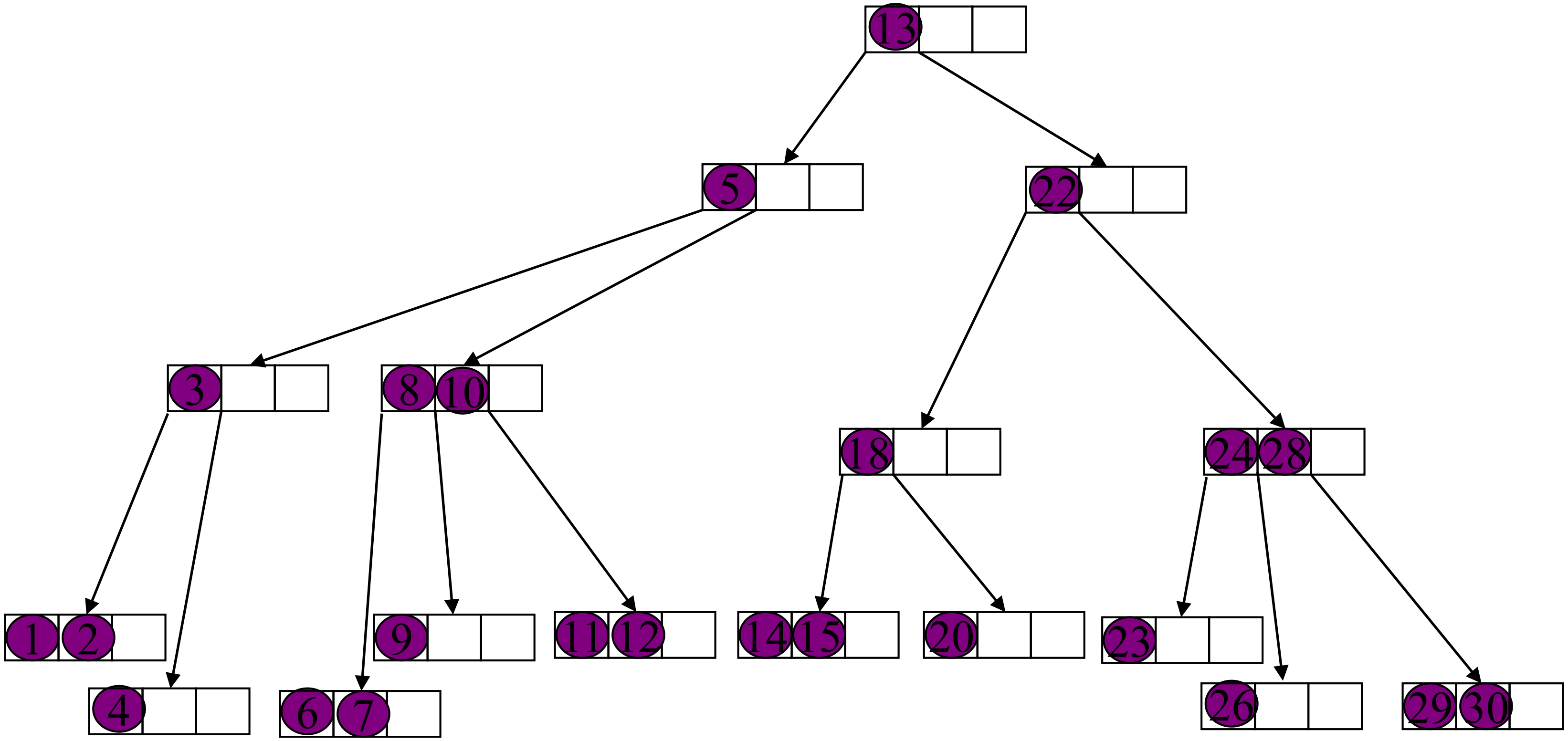
Deletion(2)

- If key to be deleted is in an internal node then we swap it with its predecessor (which is in a leaf) and then delete it.
- Delete 25



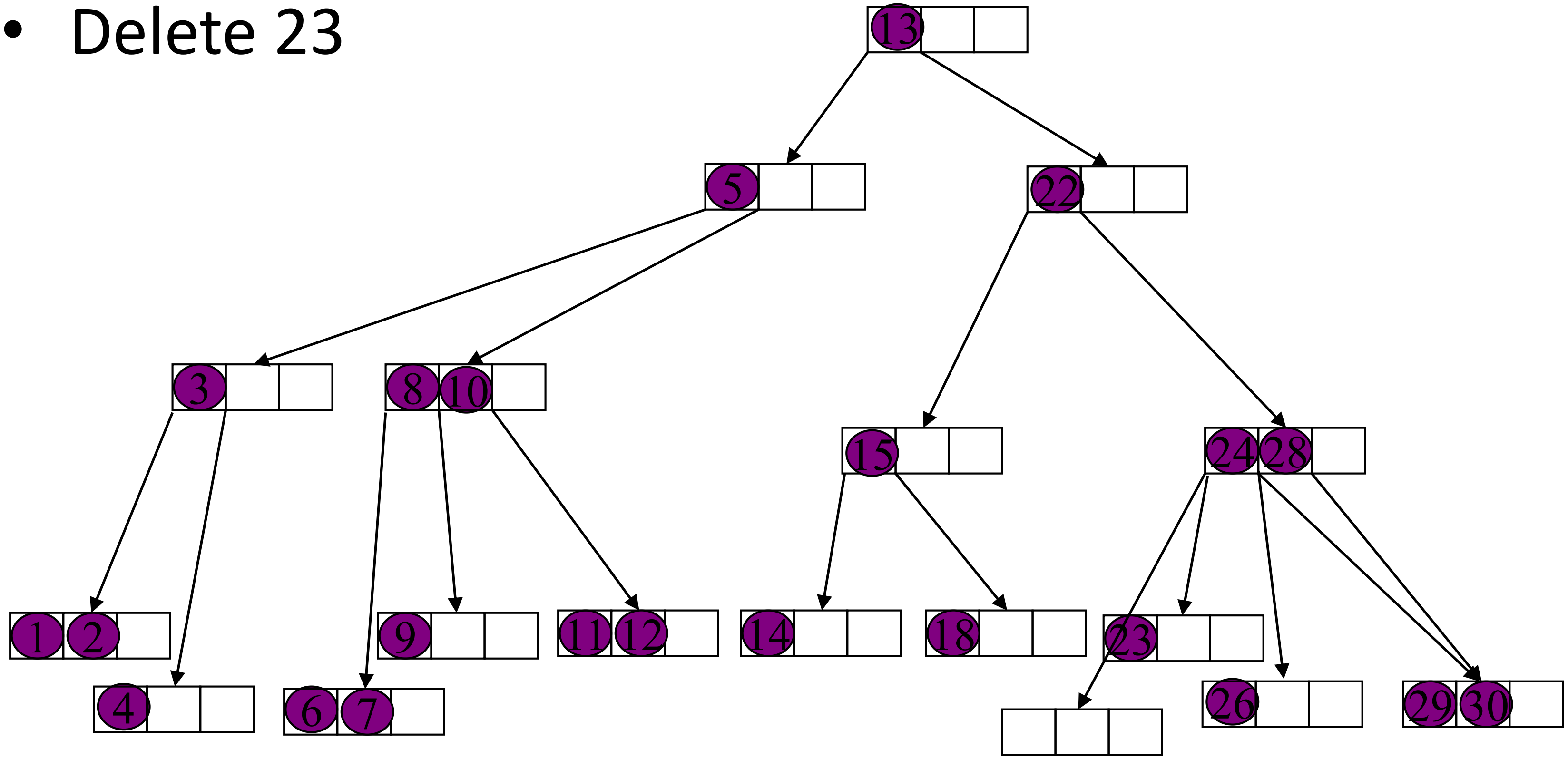
Deletion(3)

- If after deleting a key a node becomes empty then we borrow a key from its sibling.
- Delete 20



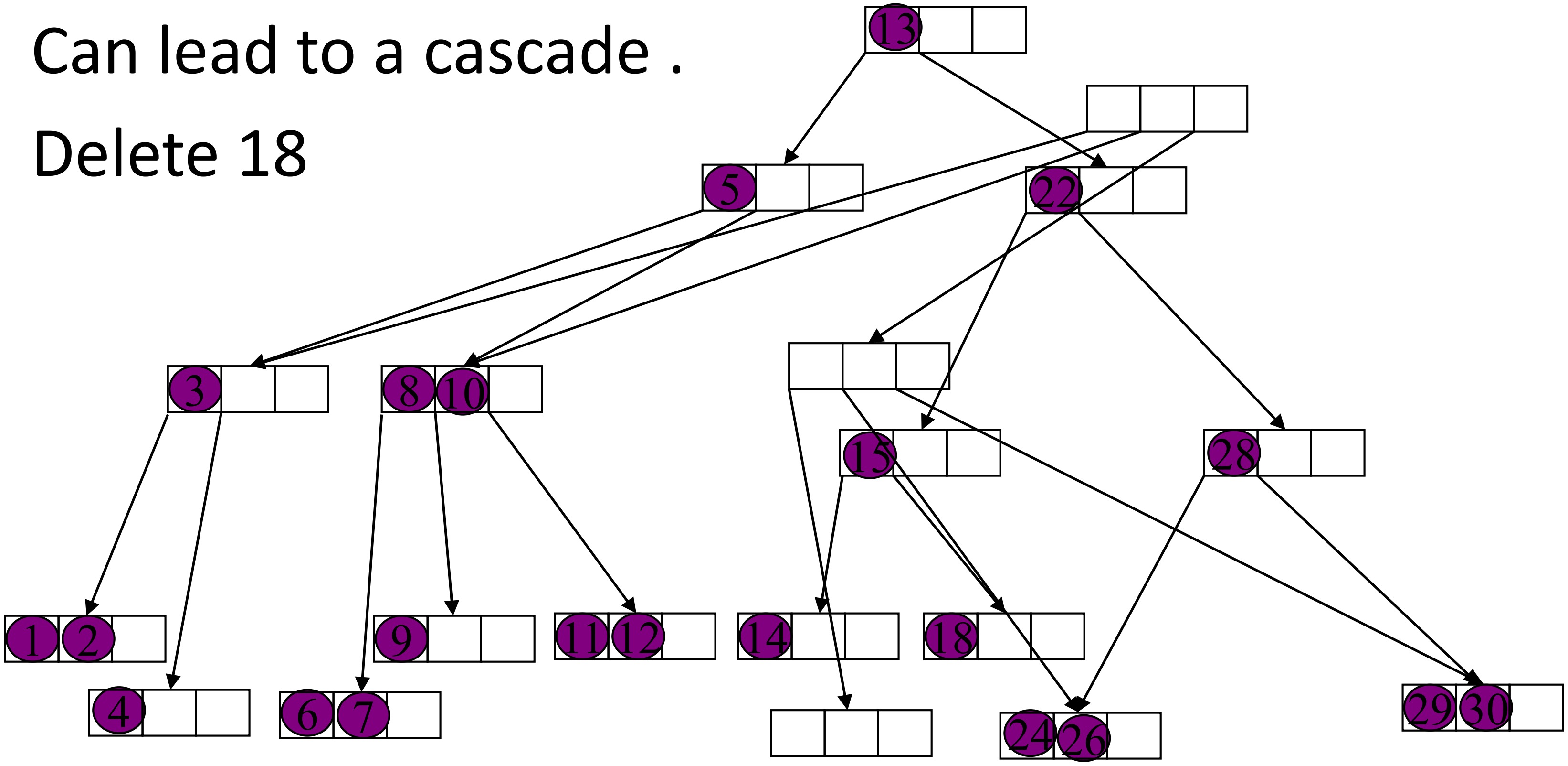
Deletion(4)

- If sibling has only one key then we merge with it.
- The key in the parent node separating these two siblings moves down into the merged node.
- Delete 23



Delete(5)

- Moving a key down from the parent corresponds to deletion in the parent node.
- The procedure is the same as for a leaf node.
- Can lead to a cascade .
- Delete 18

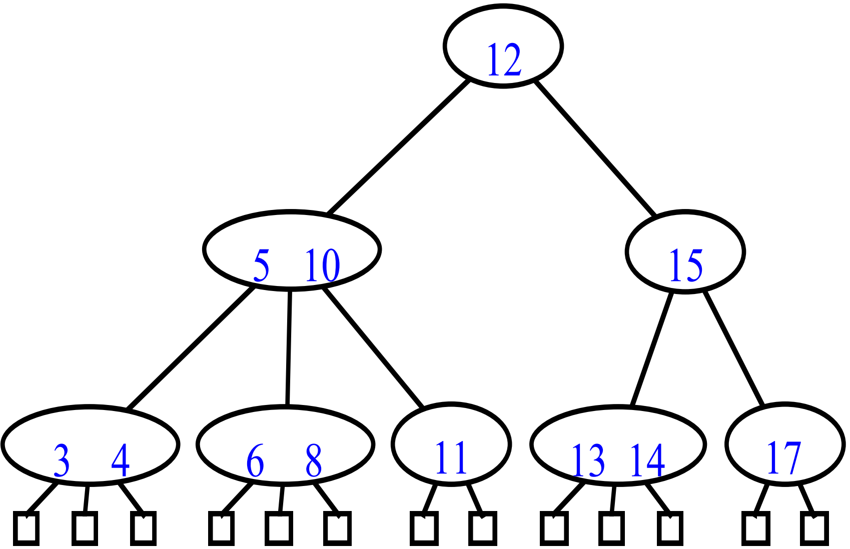


(2,4) Conclusion

- The height of a (2,4) tree is *O*(log *n*).
- Split, transfer, and merge each take *O*(1).
- Search, insertion and deletion each take *O*(log *n*) .
- Why are we doing this?
  - (2,4) trees are fun! Why else would we do it?
  - Well, there's another reason, too.
  - They can be extended to what are called B-trees.

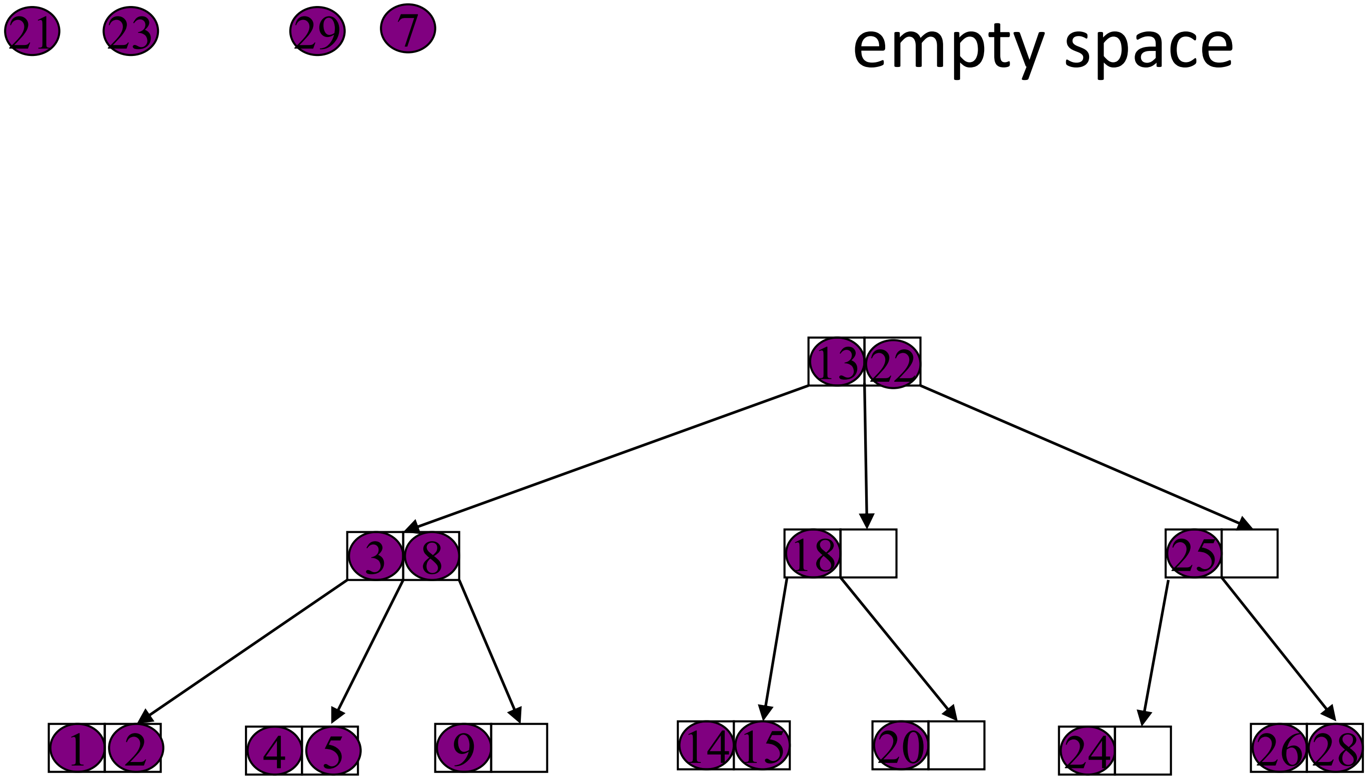
(a,b) Trees

- A multiway search tree.
- Each node has at least *a* and at most *b* children.
- Root can have less than *a* children but it has at least 2 children.
- All leaf nodes are at the same level.
- Height *h* of (a,b) tree is at least $\log_b n$ and at most $\log_a n$.

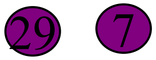


Insertion

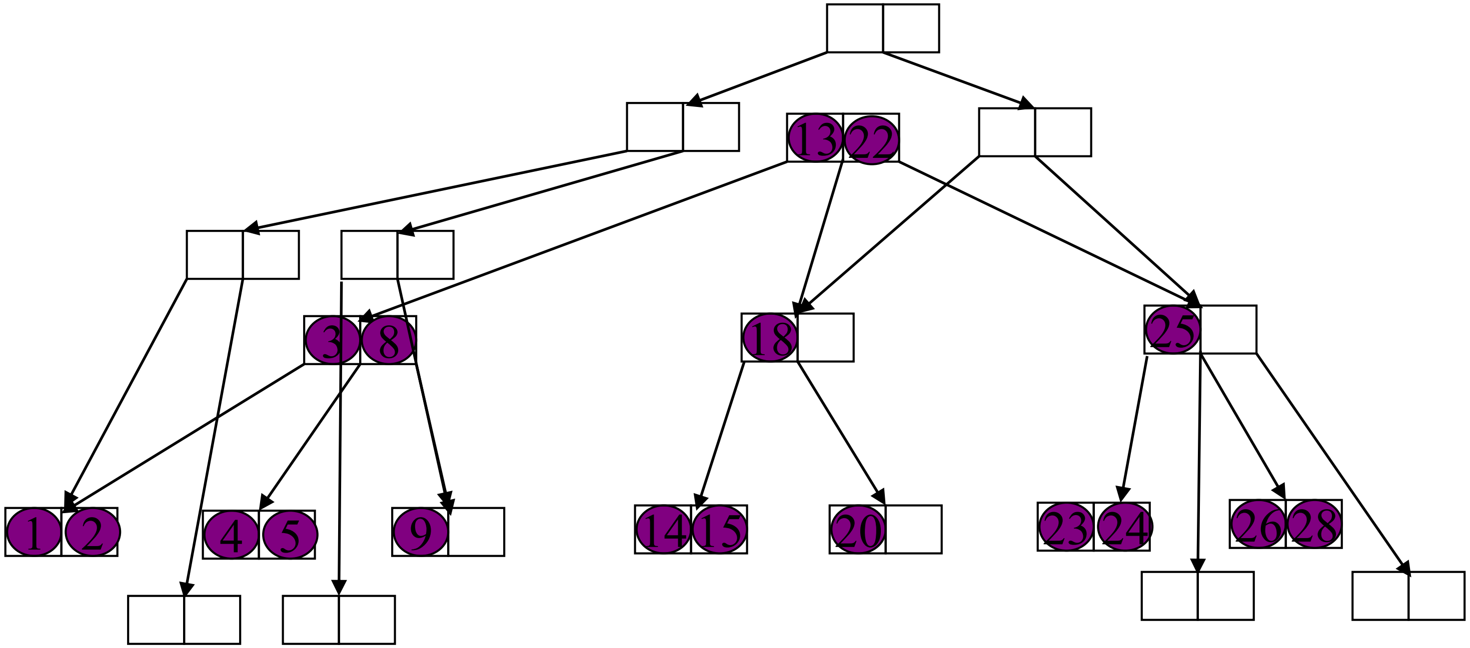
- No problem if the node has empty space



Insertion(2)



- Nodes get split if there is insufficient space.
- The median key is promoted to the parent node and inserted there



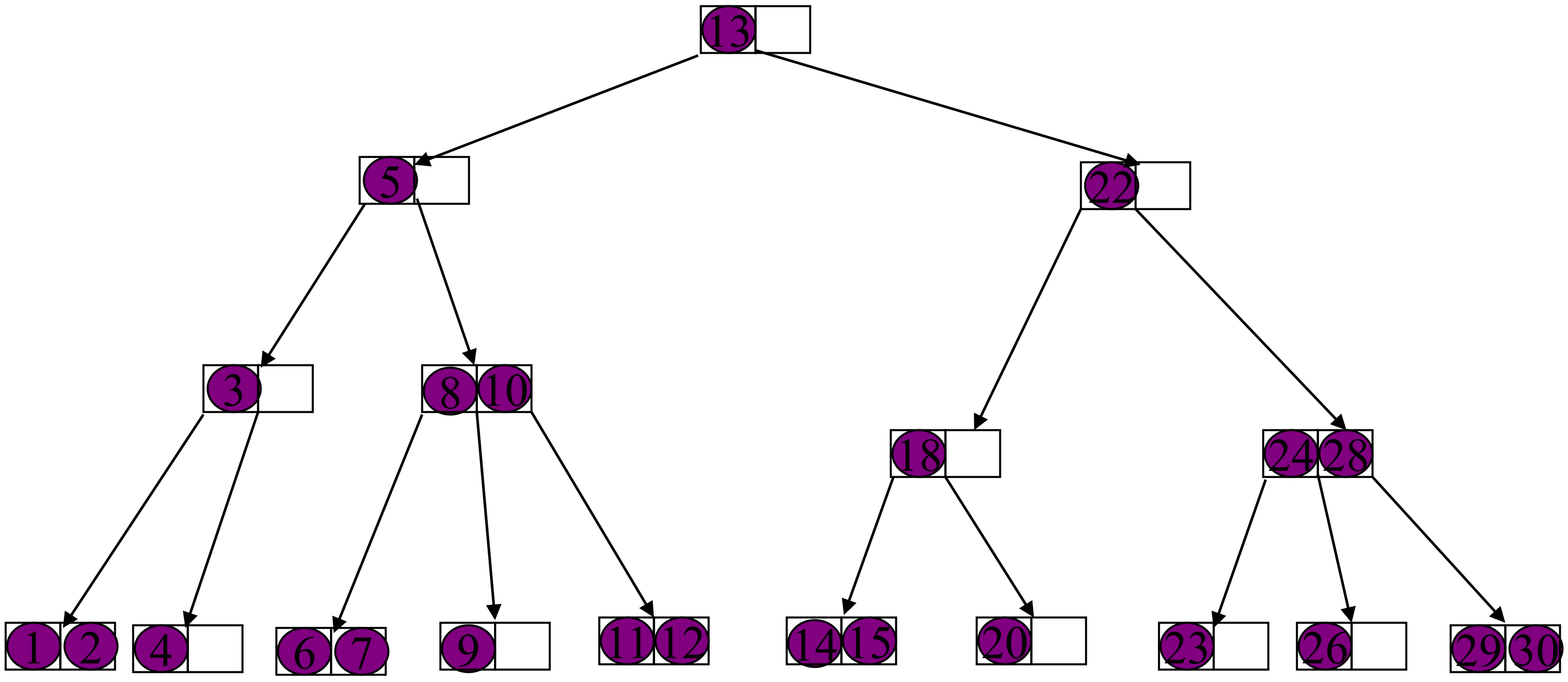
Insertion(3)

- A node is split when it has exactly *b* keys.
- One of these is promoted to the parent and the remaining are split between two nodes.
- Thus one node gets $\lceil \frac{b-1}{2} \rceil$ and the other $\lfloor \frac{b-1}{2} \rfloor$ keys.

- This implies that a-1 >= $\lfloor \frac{b-1}{2} \rfloor$

Deletion

- If after deleting a key a node becomes empty then we borrow a key from its sibling.
- Delete 20



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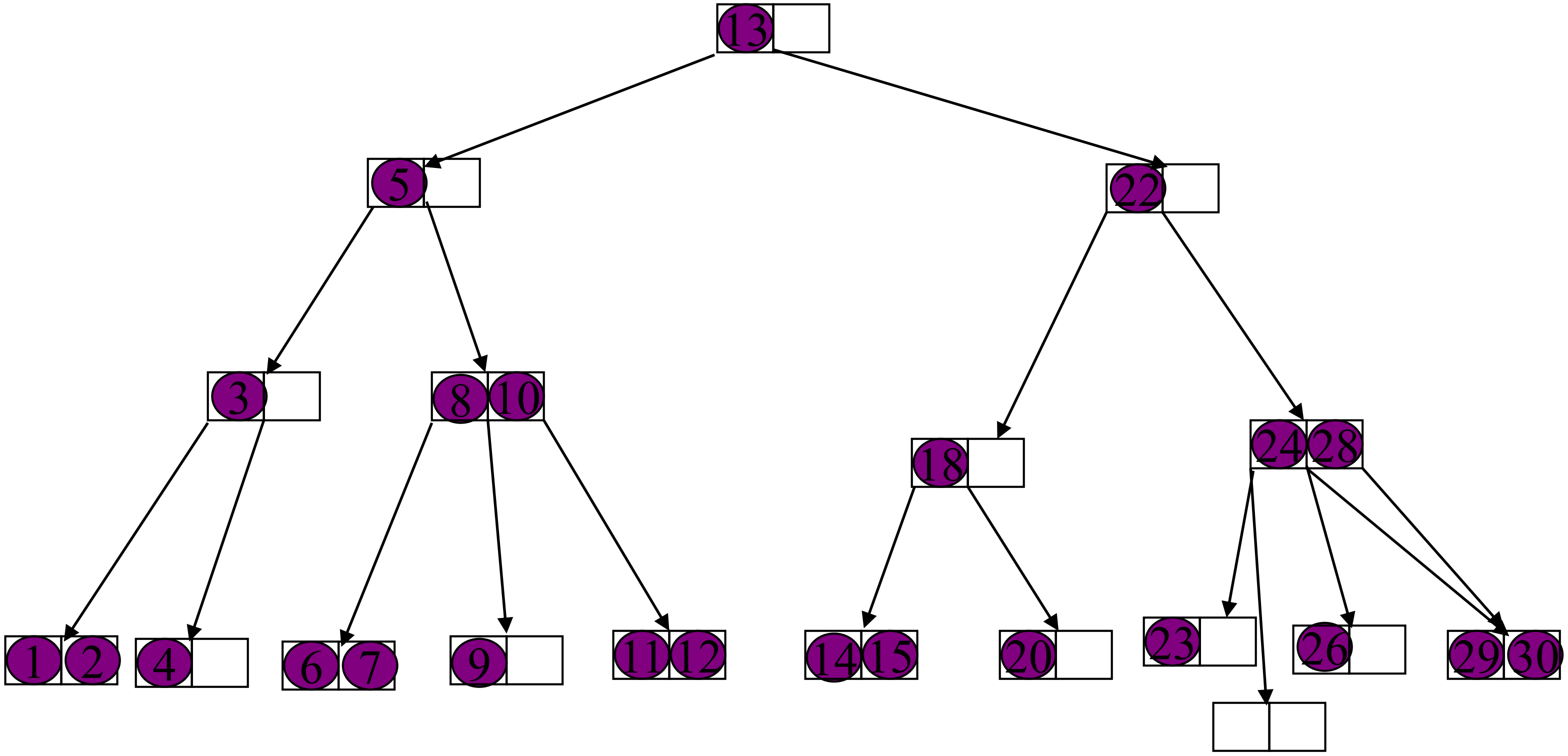
- Delete 23



Deletion(3)

- In an (a,b) tree we will merge a node with its sibling if the node has *a*-2 keys and its sibling has *a*-1 keys.
- Thus the merged node has 2(*a*-1) keys.
- This implies that 2(*a*-1) <= *b*-1 which is equivalent to *a*-1 <= $\lfloor \frac{b-1}{2} \rfloor$
- Earlier too we argued that *a*-1 <= $\lfloor \frac{b-1}{2} \rfloor$
- This implies *b* >= 2*a*-1
- For a=2, *b* >= 3

Conclusion

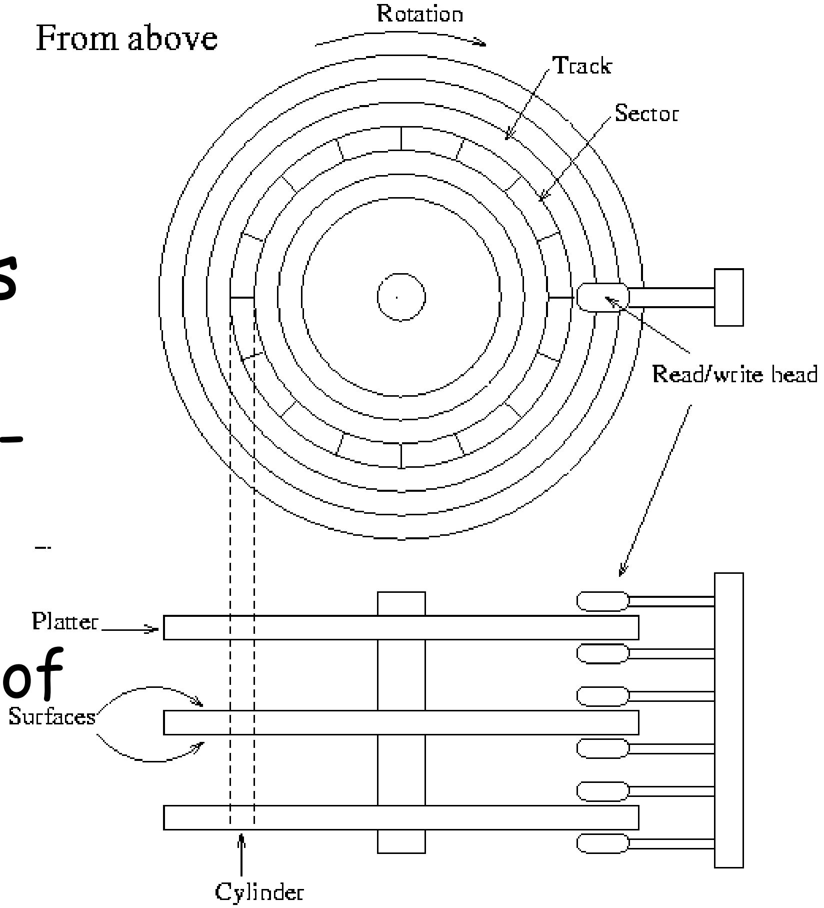
- The height of a (a,b) tree is $O(\log n)$.
- $b >= 2a-1$.
- For insertion and deletion we take time proportional to the height.

Disk Based Data Structures

- So far search trees were limited to main memory structures
  - Assumption: the dataset organized in a search tree fits in main memory (including the tree overhead)
- Counter-example: transaction data of a bank > 1 GB per day
  - use secondary storage media (punch cards, hard disks, magnetic tapes, etc.)
- Consequence: make a search tree structure secondary-storage-enabled

Hard Disks

- **Large amounts of storage, but slow access!**
- Identifying a page takes a long time (seek time plus rotational delay – 5-10ms), reading it is fast
  - – pays off to read or write data in **pages** (or blocks) of 2-16 Kb in size.



Algorithm analysis

- The running time of disk-based algorithms is measured in terms of
  - computing time (CPU)
  - number of disk accesses
    - sequential reads
    - random reads

- Regular main-memory algorithms that work one data element at a time can not be “ported” to secondary storage in a straight-forward way

Principles

- Pointers in data structures are no longer addresses in main memory but locations in files
- If *x* is a pointer to an object
  - if *x* is in main memory *key*[*x*] refers to it
  - otherwise DiskRead(*x*) reads the object from disk into main memory (DiskWrite(*x*) – writes it back to disk)

Principles (2)

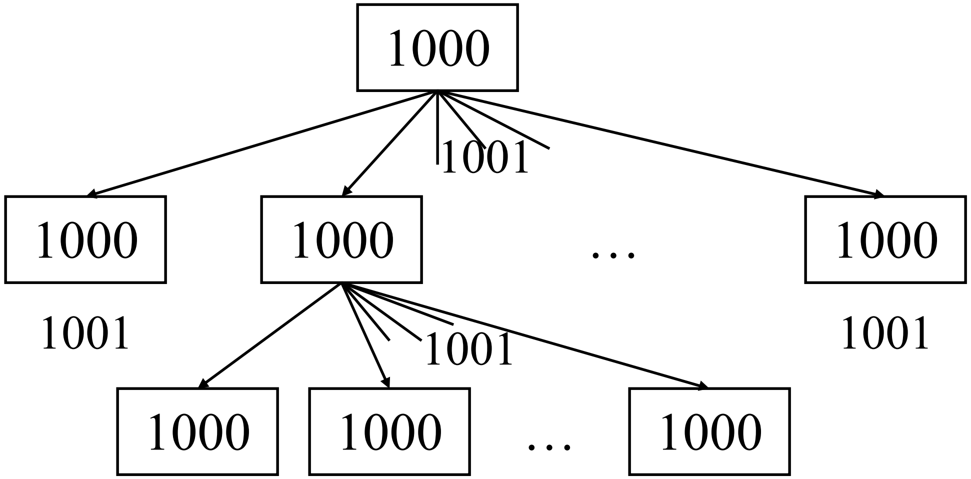
- A typical working pattern

```
…
x ← a pointer to some object
DiskRead(x)
operations that access and/or modify x
DiskWrite(x) //omitted if nothing changed
other operations, only access no modify
…
```

- Operations:
  - DiskRead(x:pointer_to_a_node)
  - DiskWrite(x:pointer_to_a_node)
  - AllocateNode():pointer_to_a_node

Binary-trees vs. B-trees

- Size of B-tree nodes is determined by the page size. One page – one node.
- A B-tree of height 2 may contain > 1 billion keys!
- Heights of Binary-tree and B-tree are logarithmic
  - B-tree: logarithm of base, e.g., 1000
  - Binary-tree: logarithm of base 2



1 node
1000 keys

1001 nodes,
1,001,000 keys

1,002,001 nodes,
1,002,001,000 keys

B-tree Definitions

- Node *x* has fields
  - $n[x]$: the number of keys of that the node
  - $\text{key}_1[x] \leq \ldots \leq \text{key}_{n[x]}[x]$: the keys in ascending order
  - $\text{leaf}[x]$: true if leaf node, false if internal node
  - if internal node, then $c_1[x], \ldots, c_{n[x]+1}[x]$: pointers to children
- Keys separate the ranges of keys in the sub-trees. If k_i is an arbitrary key in the subtree $c_i[x]$ then $k_i \leq \text{key}_i[x] \leq k_{i+1}$

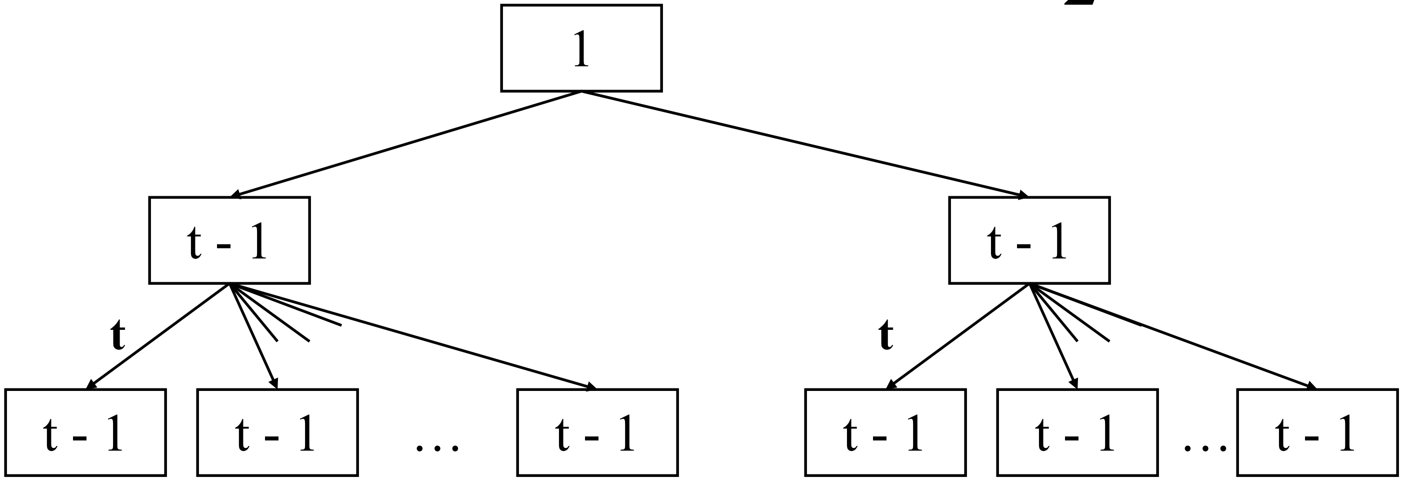
B-tree Definitions (2)

- Every leaf has the same depth
- In a B-tree of a **degree** t all nodes except the root node have between t and $2t$ children (i.e., between t–1 and $2t$–1 keys).
- The root node has between 0 and $2t$ children (i.e., between 0 and $2t$–1 keys)

Height of a B-tree

- B-tree *T* of height *h*, containing $n \geq 1$ keys and minimum degree $t \geq 2$, the following restriction on the height holds:

$$h \leq \log_t \frac{n+1}{2}$$



depth	#of nodes
0	1
1	2
2	2t

$$n \geq 1 + (t-1)\sum_{i=1}^{h} 2t^{i-1} = 2t^h - 1$$

B-tree Operations

- An implementation needs to suport the following B-tree operations
  - **Searching** (simple)
  - **Creating** an empty tree (trivial)
  - **Insertion** (complex)
  - **Deletion** (complex)

Searching

- Straightforward generalization of a binary tree search

```
BTreeSearch(x,k)
i ← 1
while i ≤ n[x] and k > $key_i[x]$
   i ← i+1
if i ≤ n[x] and k = $key_i[x]$ then
   return(x,i)
if leaf[x] then
   return NIL
   else DiskRead($c_i[x]$)
        return BTtreeSearch($c_i[x]$,k)
```

Creating an Empty Tree

- Empty B-tree = create a root & write it to disk!

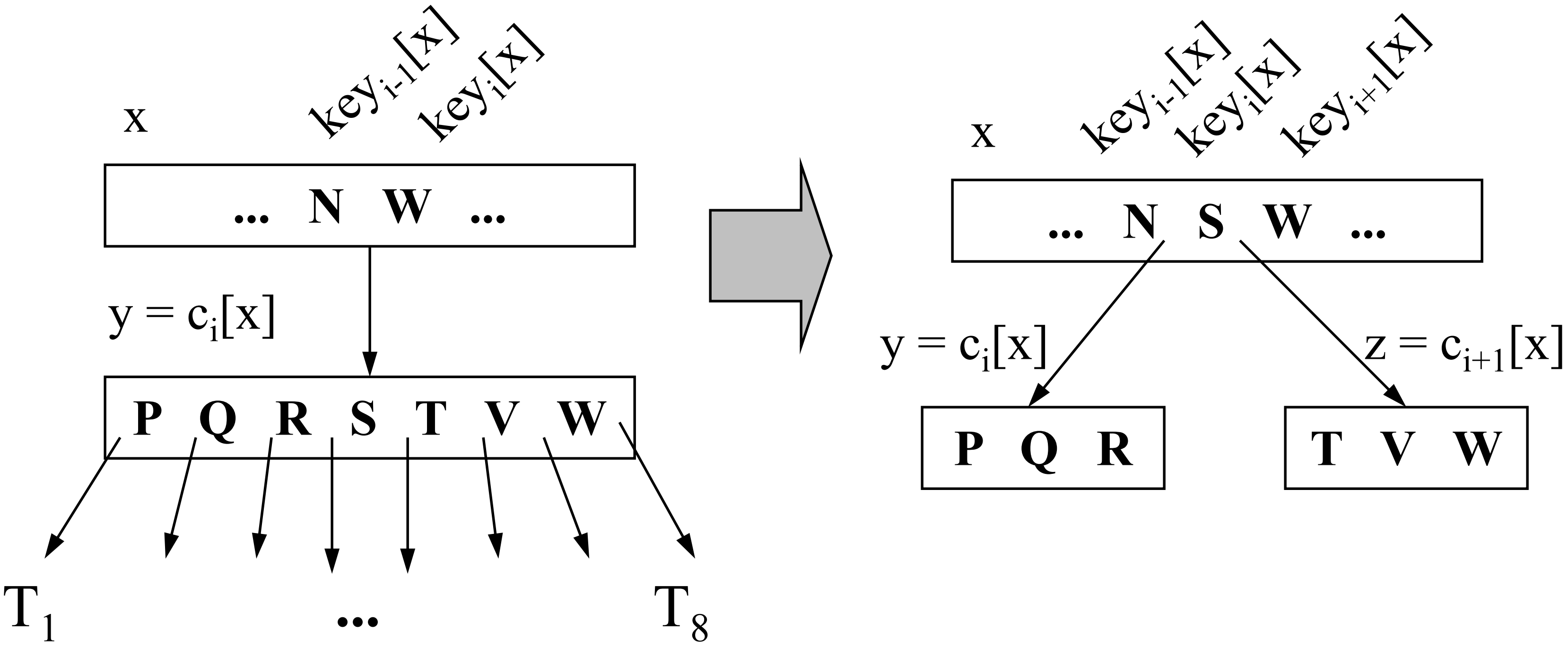
```
BTreeCreate(T)
x ← AllocateNode();
leaf[x] ← TRUE;
n[x] ← 0;
DiskWrite(x);
root[T] ← x
```

Splitting Nodes

- Nodes fill up and reach their maximum capacity $2t - 1$
- Before we can insert a new key, we have to “make room,” i.e., split nodes

Splitting Nodes (2)

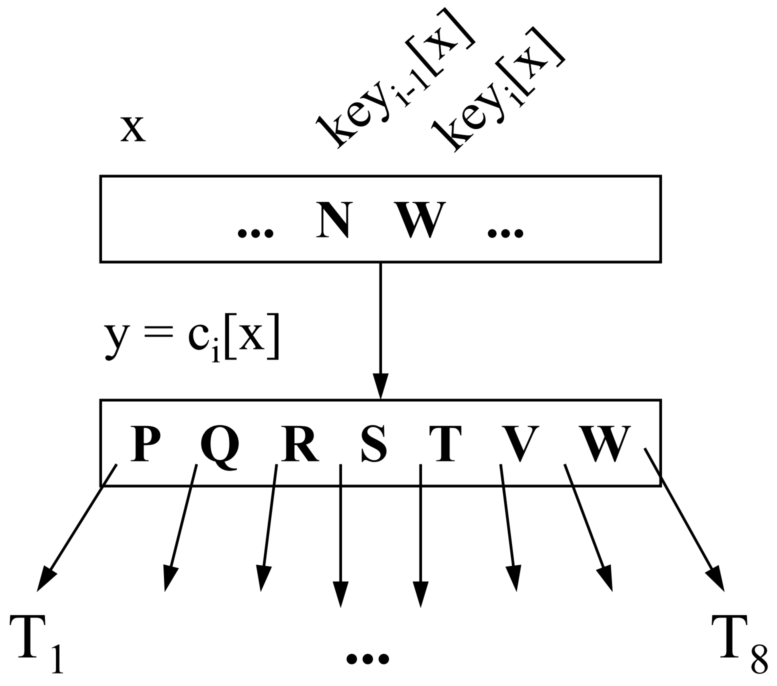
- Result: one key of *x* moves up to parent + 2 nodes with *t*-1 keys



Splitting Nodes (2)

```
BTreeSplitChild(x,i,y)
  z ← AllocateNode()
  leaf[z] ← leaf[y]
  n[z] ← t-1
  for j ← 1 to t-1
     $key_j[z]$ ← $key_{j+t}[y]$
  if not leaf[y] then
     for j ← 1 to t
        $c_j[z]$ ← $c_{j+t}[y]$
  n[y] ← t-1
  for j ← n[x]+1 downto i+1
     $c_{j+1}[x]$ ← $c_j[x]$
  $c_{i+1}[x]$ ← z
  for j ← n[x] downto i
     $key_{j+1}[x]$ ← $key_j[x]$
  $key_i[x]$ ← $key_t[y]$
  n[x] ← n[x]+1
  DiskWrite(y)
  DiskWrite(z)
  DiskWrite(x)
```

x: parent node
y: node to be split and child of x
i: index in x
z: new node



Split: Running Time

- A local operation that does not traverse the tree
- $\Theta(t)$ CPU-time, since two loops run t times
- 3 I/Os

Inserting Keys

- Done recursively, by starting from the root and recursively traversing down the tree to the leaf level
- Before descending to a lower level in the tree, make sure that the node contains $< 2t - 1$ keys:
  - so that if we split a node in a lower level we will have space to include a new key

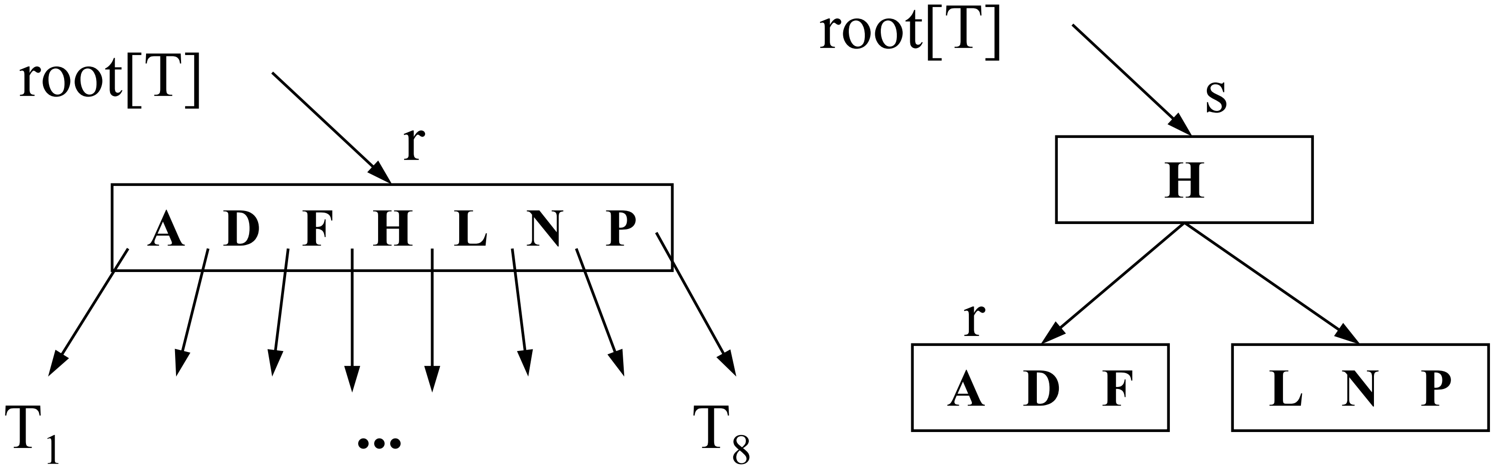
Inserting Keys (2)

- Special case: root is full (BtreeInsert)

```
BTreeInsert(T)
 r ← root[T]
 if n[r] = 2t - 1 then
    s ← AllocateNode()
    root[T] ← s
    leaf[s] ← FALSE
    n[s] ← 0
    $c_1$[s] ← r
    BTreeSplitChild(s,1,r)
    BTreeInsertNonFull(s,k)
 else BTreeInsertNonFull(r,k)
```

Splitting the Root

- Splitting the root requires the creation of a new root



- The tree grows at the top instead of the bottom

Inserting Keys

- BtreeNonFull tries to insert a key *k* into a node *x*, which is **assumed to be non-full** when the procedure is called
- BTreeInsert and the recursion in BTreeInsertNonFull guarantee that this assumption is true!

Inserting Keys: Pseudo Code

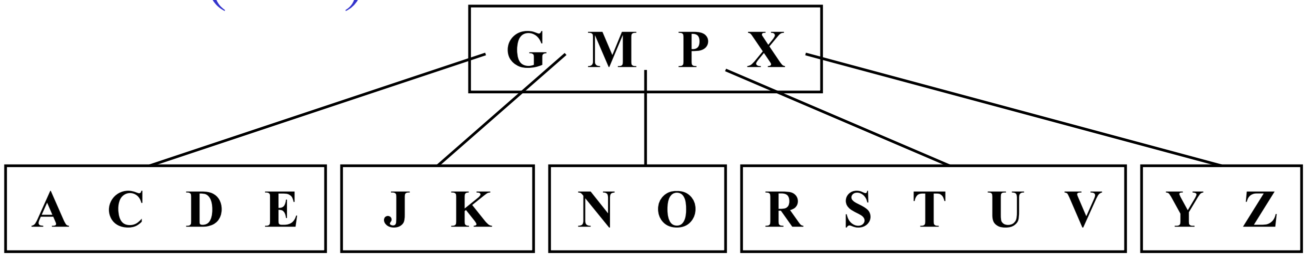
```
BTreeInsertNonFull(x,k)
i ← n[x]
if leaf[x] then
    while i ≥ 1 and k < $key_i$[x]
        $key_{i+1}$[x] ← $key_i$[x]
        i ← i - 1
    $key_{i+1}$[x] ← k
    n[x] ← n[x] + 1
    DiskWrite(x)
else while i ≥ 1 and k < $key_i$[x]
        i ← i - 1
    i ← i + 1
    DiskRead $c_i$[x]
    if n[$c_i$[x]] = 2t - 1 then
        BTreeSplitChild(x,i,$c_i$[x])
        if k > $key_i$[x] then
            i ← i + 1
    BTreeInsertNonFull($c_i$[x],k)
```

leaf insertion

internal node:
traversing tree

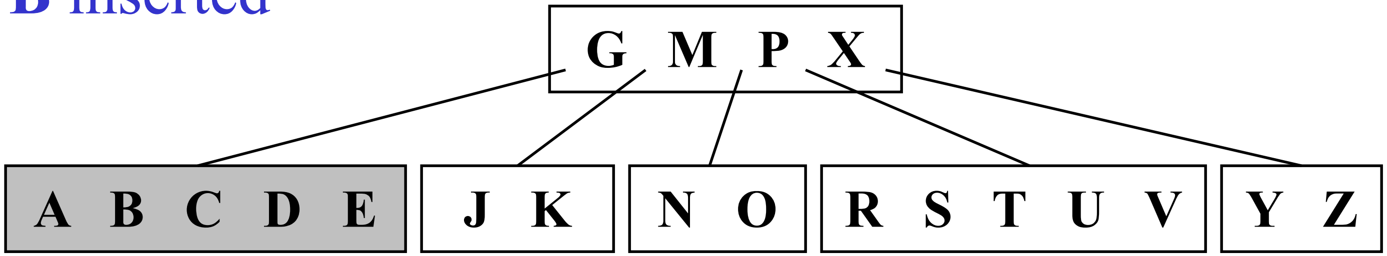
Insertion: Example

initial tree (t = 3)


G M P X

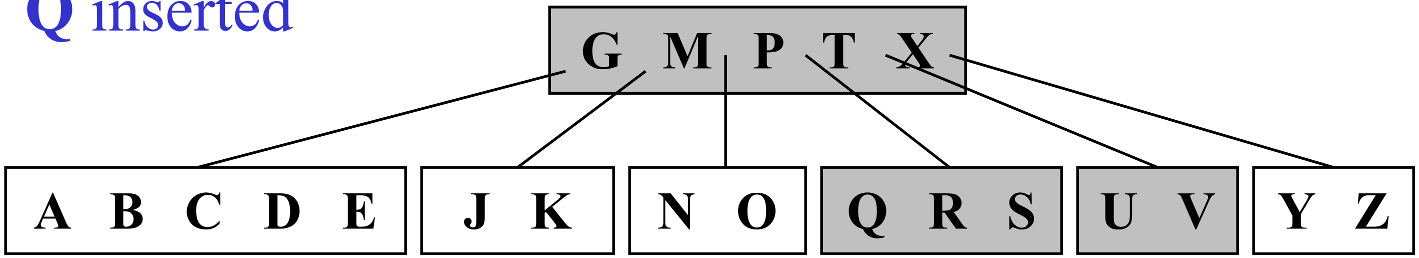
A C D E | J K | N O | R S T U V | Y Z

B inserted


G M P X

A B C D E | J K | N O | R S T U V | Y Z

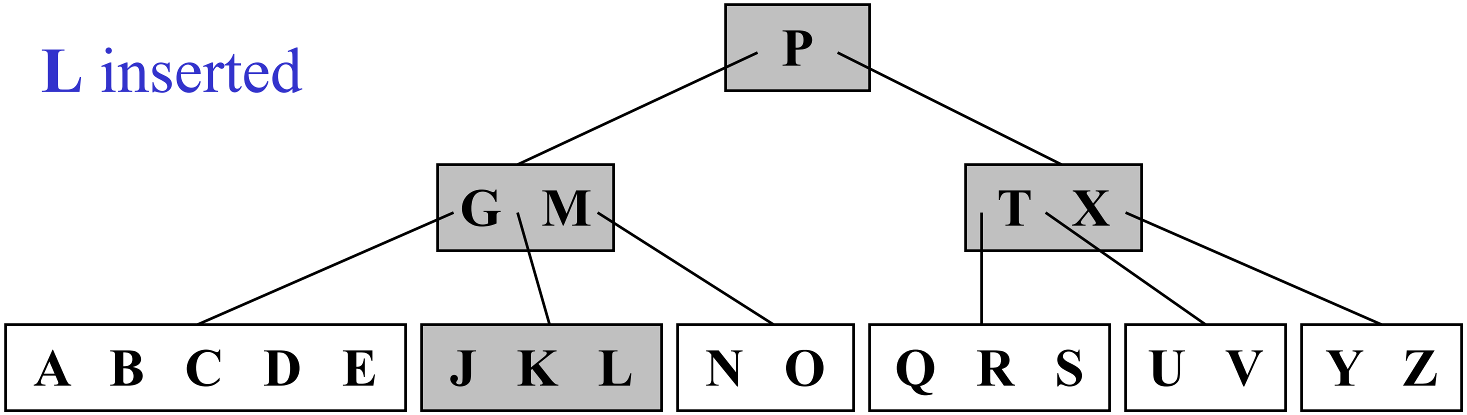
Q inserted


G M P T X

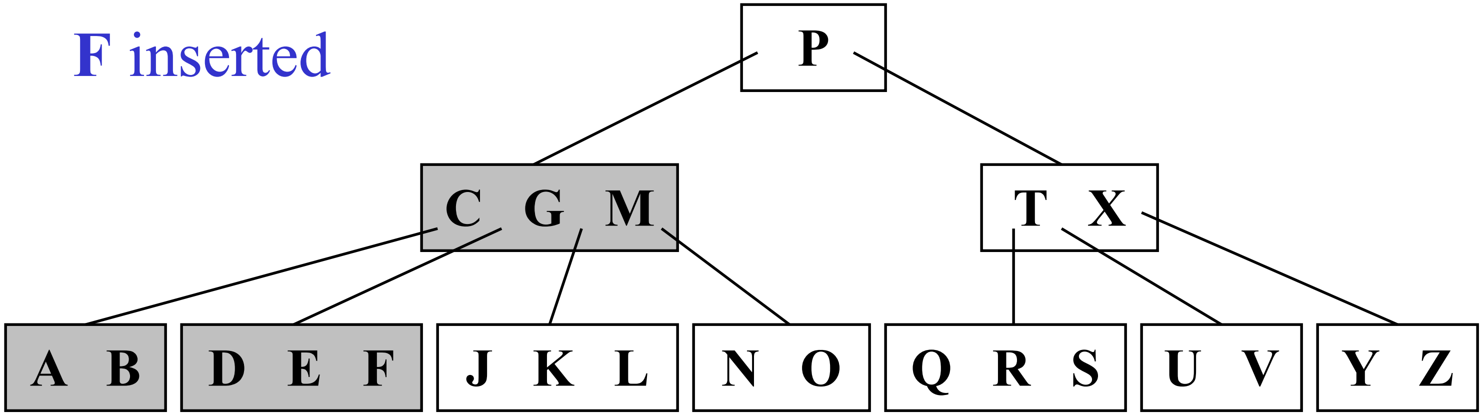
A B C D E | J K | N O | Q R S | U V | Y Z

Insertion: Example (2)

L inserted



F inserted



Insertion: Running Time

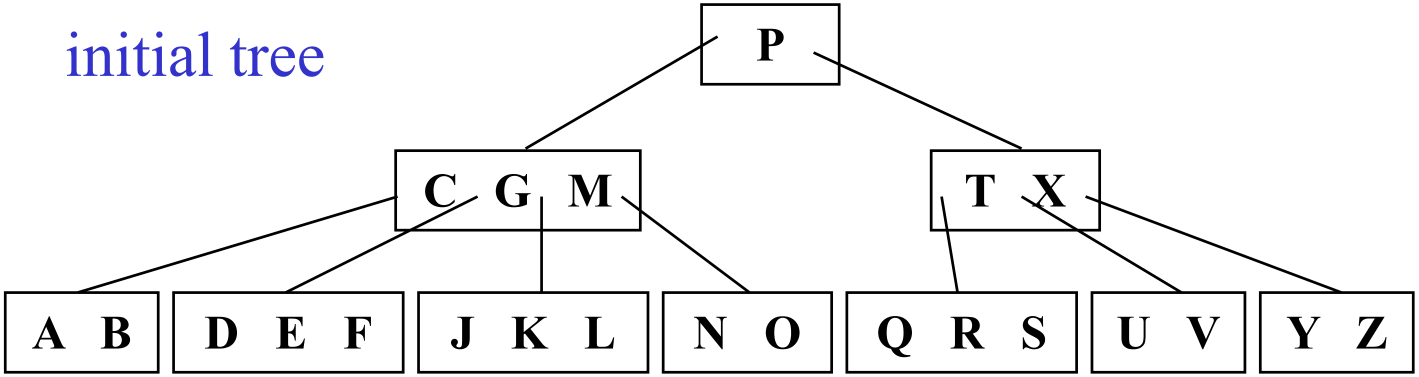
- Disk I/O: $O(h)$, since only $O(1)$ disk accesses are performed during recursive calls of BTreeInsertNonFull
- CPU: $O(th) = O(t \log_t n)$
- At any given time there are $O(1)$ number of disk pages in main memory

Deleting Keys

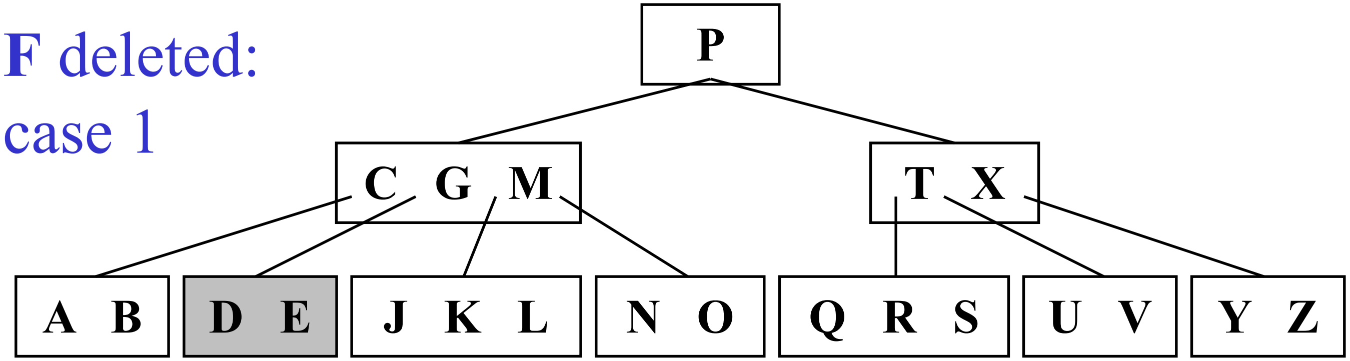
- Done recursively, by starting from the root and recursively traversing down the tree to the leaf level
- Before descending to a lower level in the tree, make sure that the node contains $\geq t$ keys (cf. insertion $< 2t - 1$ keys)
- BtreeDelete distinguishes three different stages/scenarios for deletion
  - Case 1: key *k* found in leaf node
  - Case 2: key *k* found in internal node
  - Case 3: key *k* suspected in lower level node

Deleting Keys (2)

initial tree



F deleted:
case 1

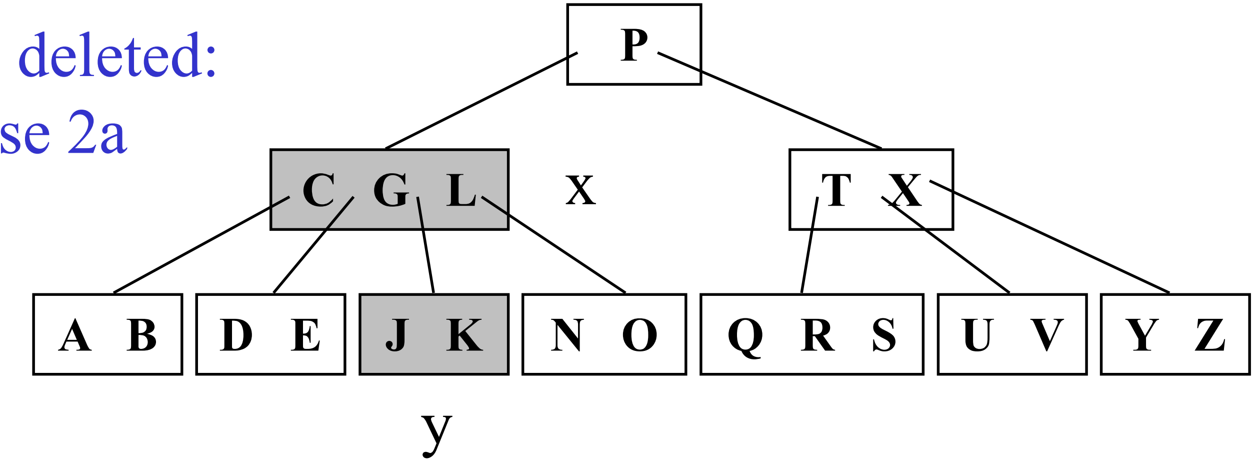


- Case 1: If the key *k* is in node *x*, and *x* is a leaf, delete *k* from *x*

Deleting Keys (3)

- Case 2: If the key *k* is in node *x*, and *x* is not a leaf, delete *k* from *x*
  - a) If the child *y* that precedes *k* in node *x* has at least *t* keys, then find the predecessor *k'* of *k* in the sub-tree rooted at *y*. Recursively delete *k'*, and replace *k* with *k'* in *x*.
  - b) Symmetrically for successor node *z*

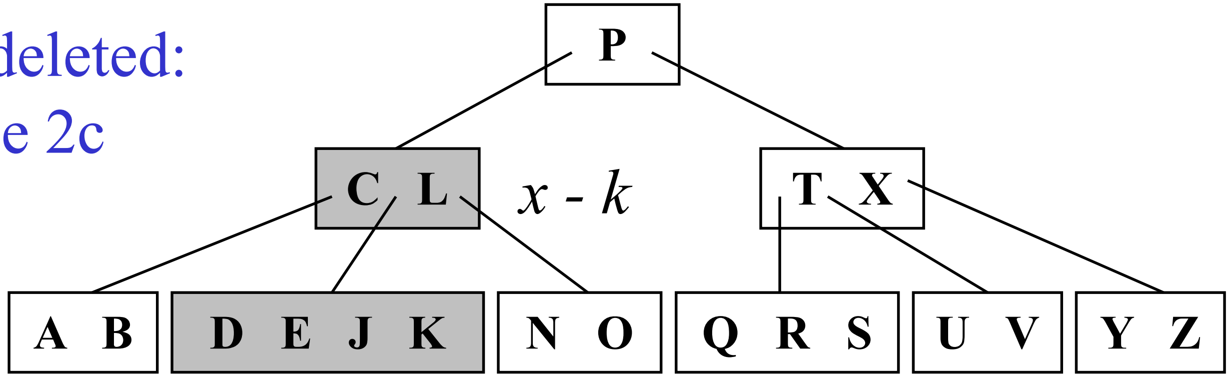
M deleted:
case 2a



Deleting Keys (4)

- If both *y* and *z* have only *t* –1 keys, ***merge*** *k* with the contents of *z* into *y*, so that *x* loses both *k* and the pointers to *z*, and *y* now contains 2*t* – 1 keys. Free *z* and recursively delete *k* from *y*.

G deleted:
case 2c



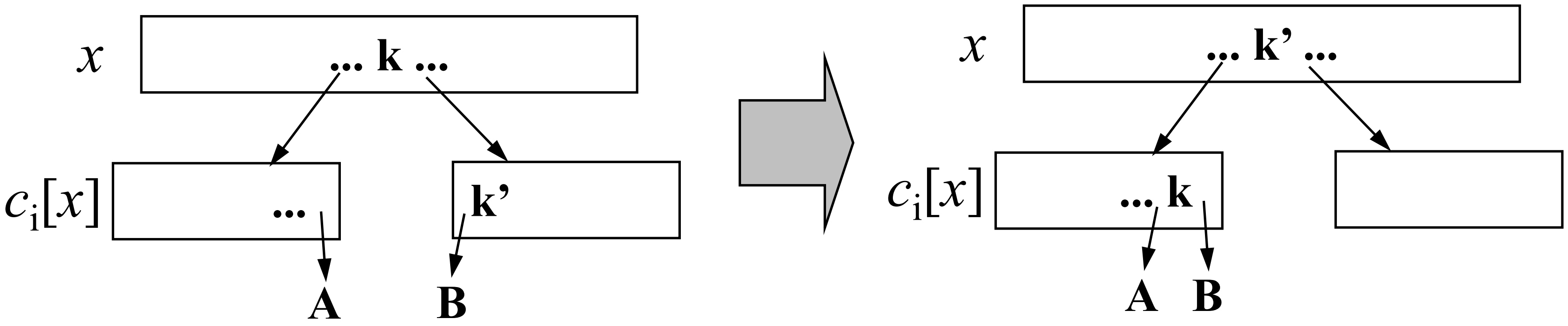
$x - k$

$y = \mathrm{y}+k + z - k$

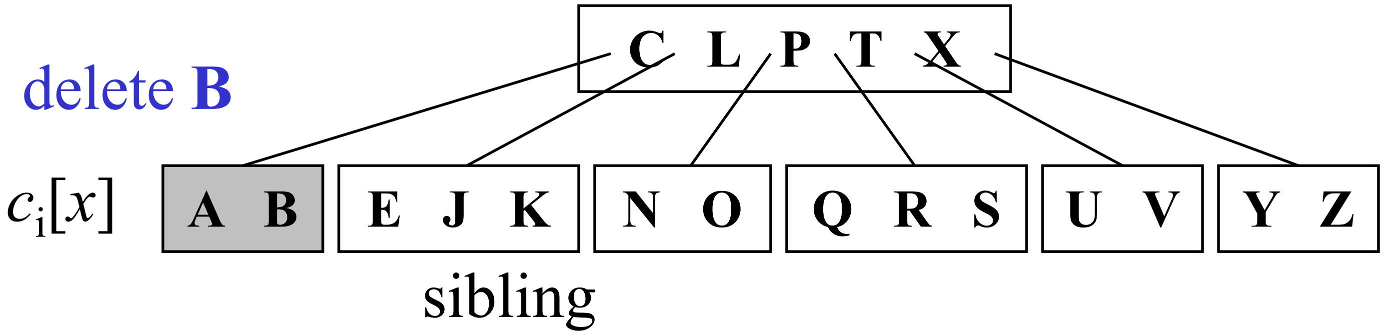
Deleting Keys - Distribution

- Descending down the tree: if *k* not found in current node *x*, find the sub-tree $c_i[x]$ that has to contain k.
- If $c_i[x]$ has only *t* – 1 keys take action to ensure that we descent to a node of size at least *t.*
- We can encounter two cases.
  - If $c_i[x]$ has only *t*-1 keys, but a sibling with at least *t* keys, give $c_i[x]$ an extra key by moving a key from *x* to $c_i[x]$, moving a key from $c_i[x]$'s immediate left and right sibling up into *x*, and moving the appropriate child from the sibling into $c_i[x]$ - ***distribution***

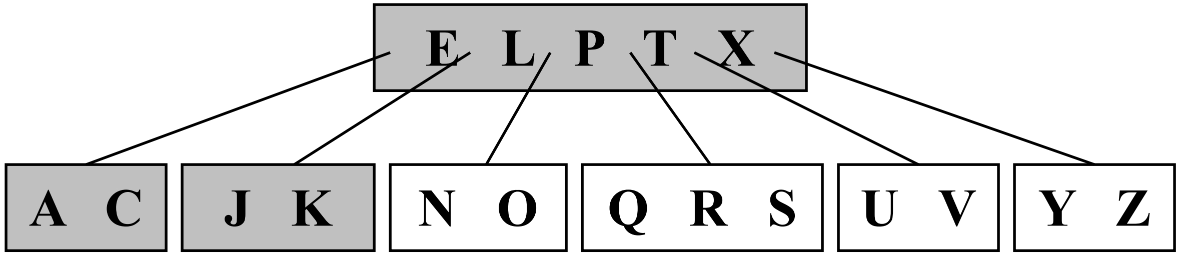
Deleting Keys – Distribution(2)



delete B

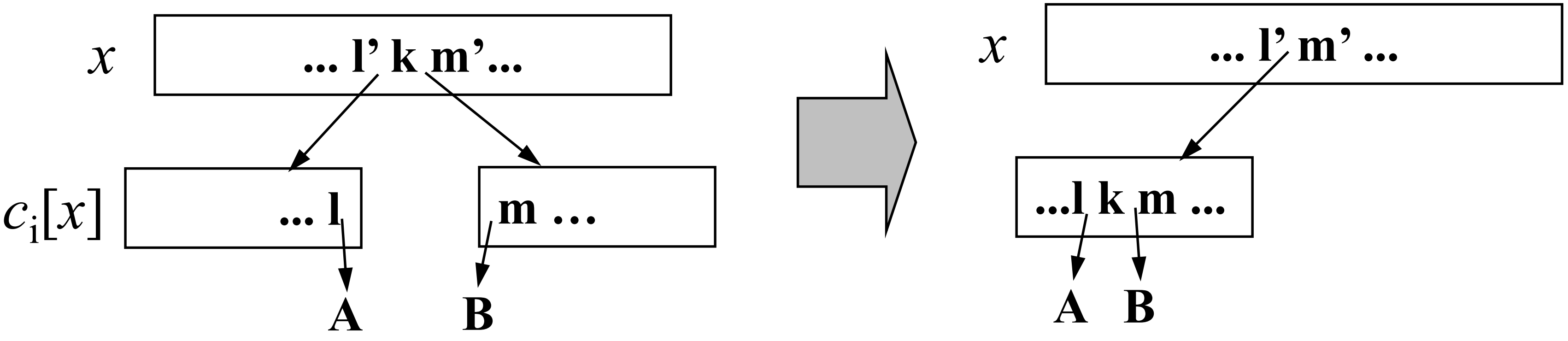


B deleted:



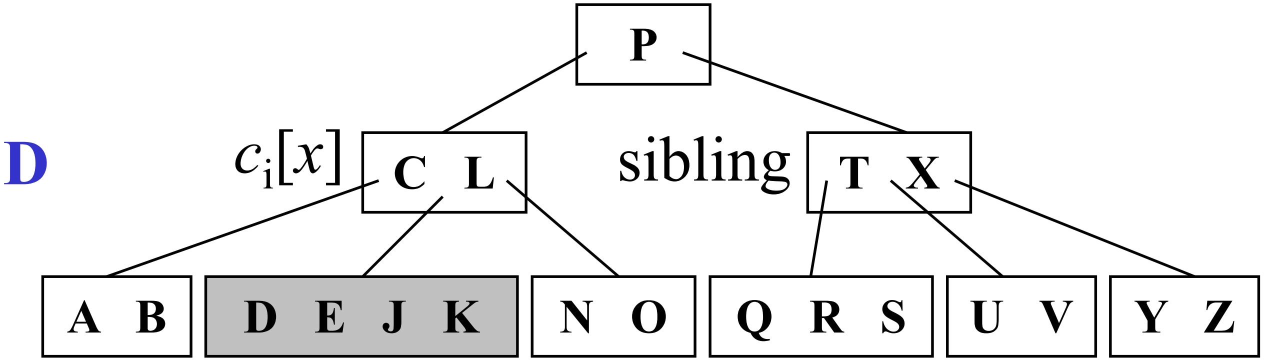
Deleting Keys - Merging

- If $c_i[x]$ and both of $c_i[x]$'s siblings have t – 1 keys, ***merge*** c_i with one sibling, which involves moving a key from *x* down into the new merged node to become the median key for that node

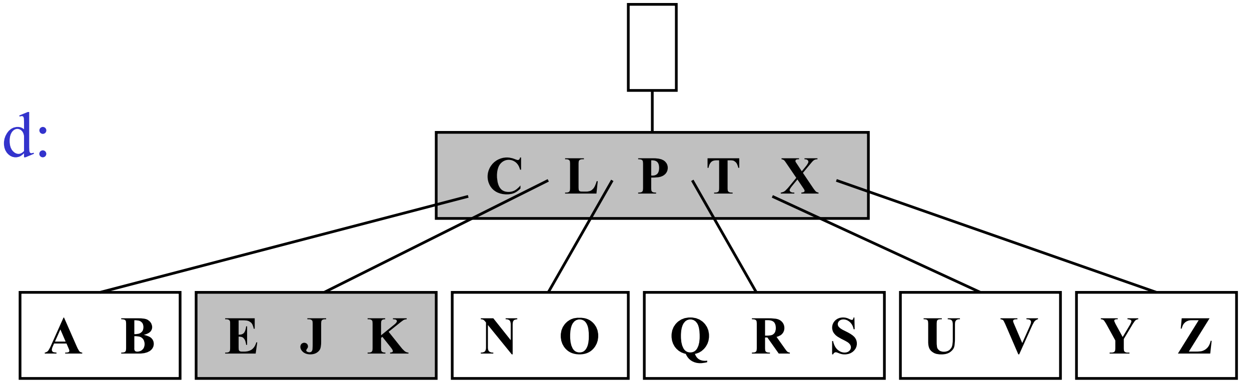


Deleting Keys – Merging (2)

delete **D**



D deleted:



tree shrinks in height

Deletion: Running Time

- Most of the keys are in the leaf, thus deletion most often occurs there!
- In this case deletion happens in one downward pass to the leaf level of the tree
- Deletion from an internal node might require “backing up” (case 2)
- Disk I/O: O(h), since only O(1) disk operations are produced during recursive calls
- CPU: $O(th) = O(t \log_t n)$

Two-pass vs One pass Operations

- Two pass simpler to implement
- One pass saves time in traversing the tree from root to leaf twice, but may cause more splits/merges than one pass.