Trees
COL 106

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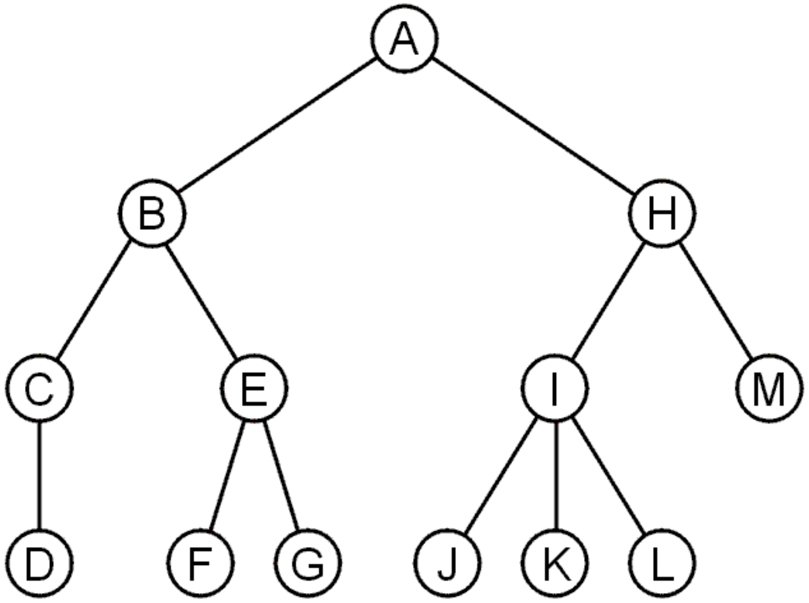
Acknowledgement :Many slides are courtesy
Douglas Harder, UWaterloo

Trees

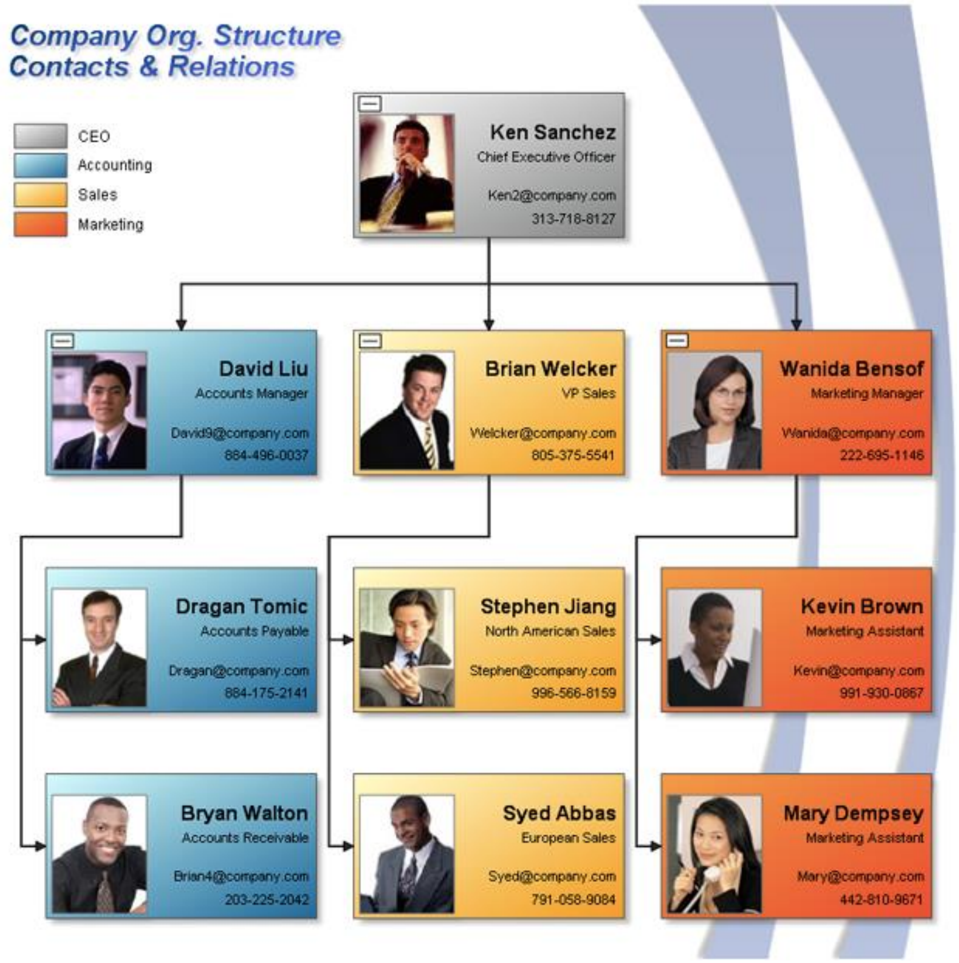
A rooted tree data structure stores information in *nodes*

- Similar to linked lists:
  - There is a first node, or *root*
  - Each node has variable number of references to successors (children)
  - Each node, other than the root, has exactly one node as its predecessor (or parent)

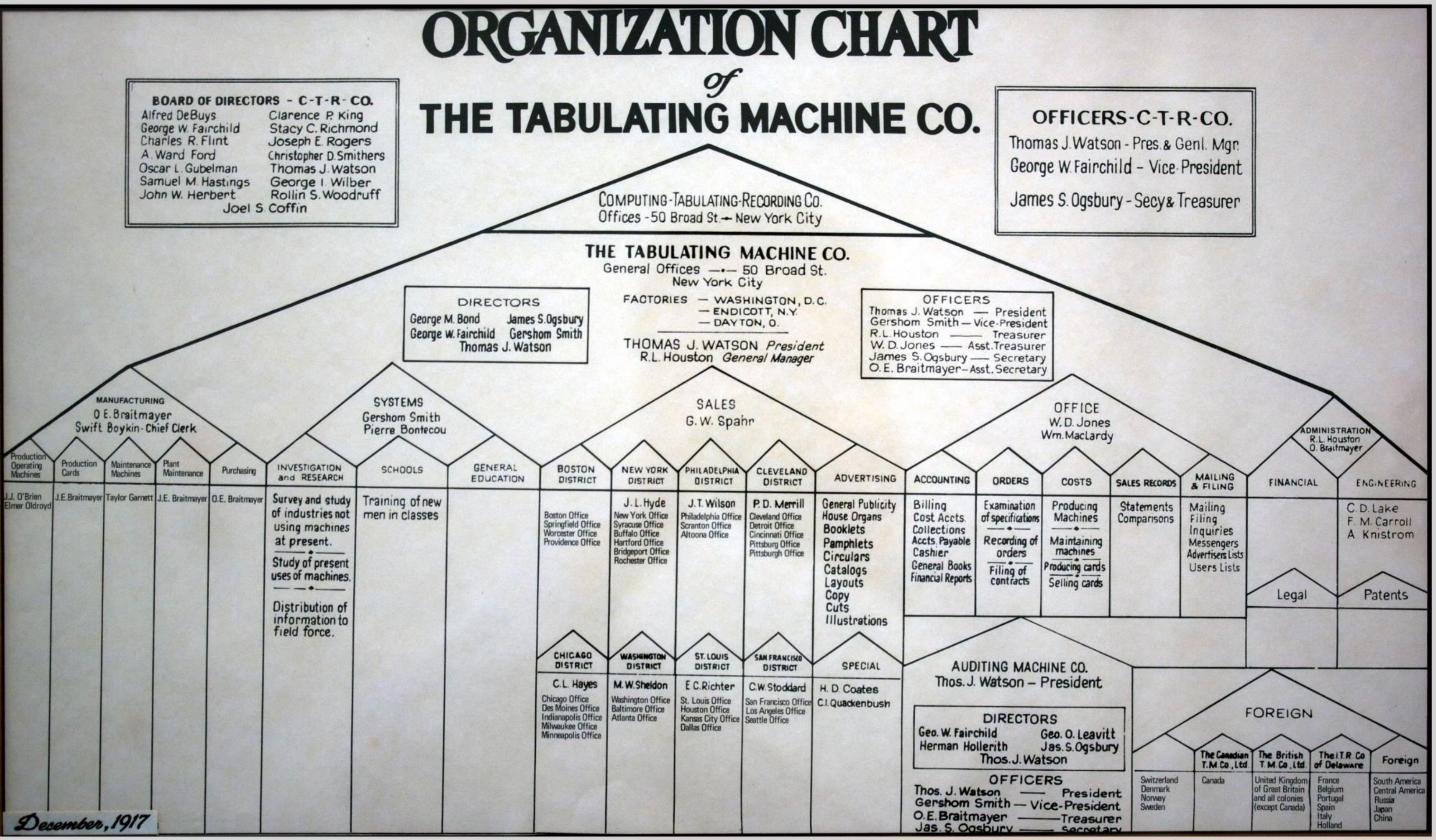


What are trees suitable for ?

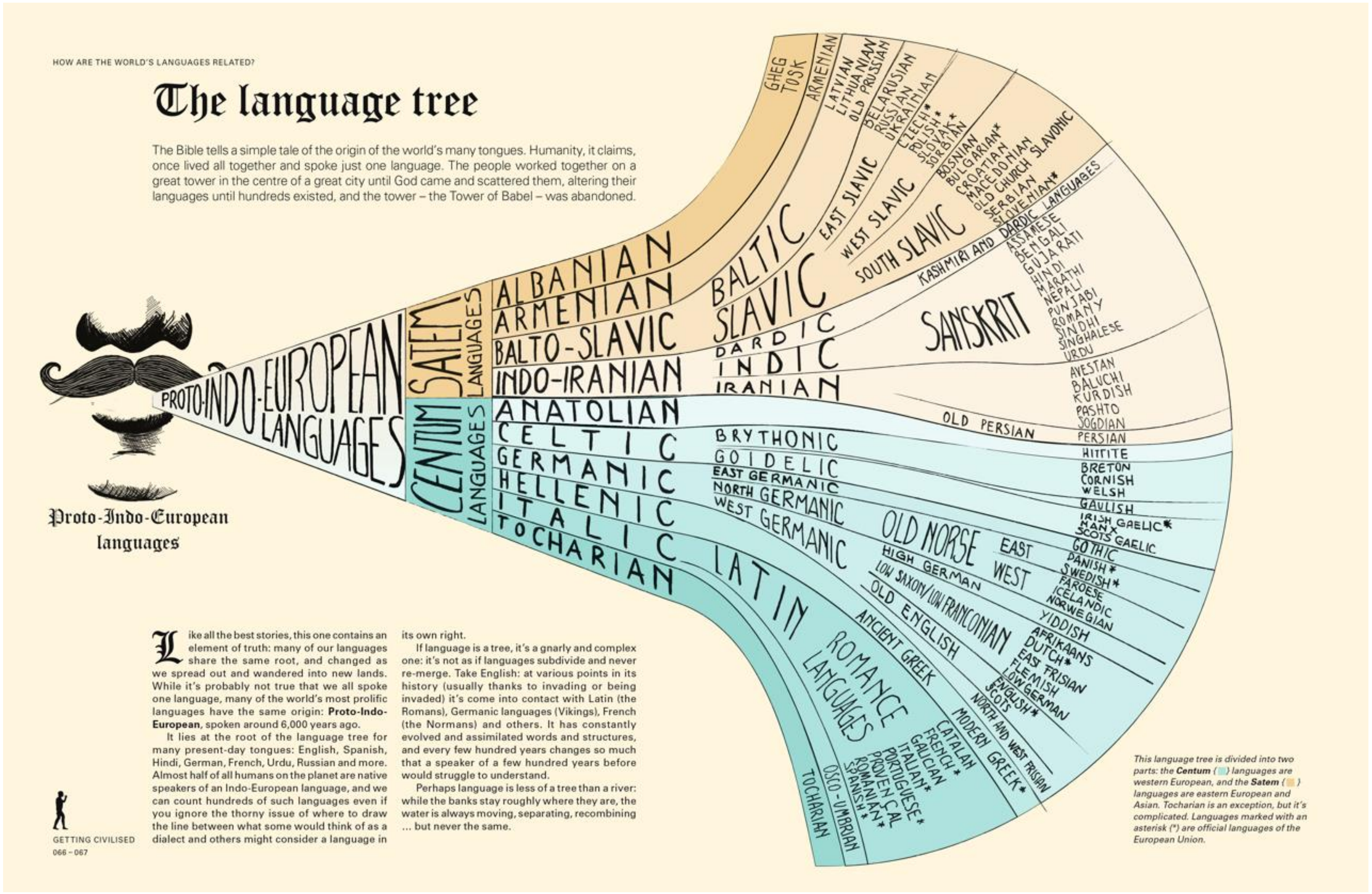
To store hierarchy of people



To store organization of departments



To capture the evolution of languages


HOW ARE THE WORLD'S LANGUAGES RELATED?

The language tree

The Bible tells a simple tale of the origin of the world's many tongues. Humanity, it claims, once lived all together and spoke just one language. The people worked together on a great tower in the centre of a great city until God came and scattered them, altering their languages until hundreds existed, and the tower – the Tower of Babel – was abandoned.

Like all the best stories, this one contains an element of truth: many of our languages share the same root, and changed as we spread out and wandered into new lands. While it's probably not true that we all spoke one language, many of the world's most prolific languages have the same origin: **Proto-Indo-European**, spoken around 6,000 years ago.

It lies at the root of the language tree for many present-day tongues: English, Spanish, Hindi, German, French, Urdu, Russian and more. Almost half of all humans on the planet are native speakers of an Indo-European language, and we can count hundreds of such languages even if you ignore the thorny issue of where to draw the line between what some would think of as a dialect and others might consider a language in its own right.

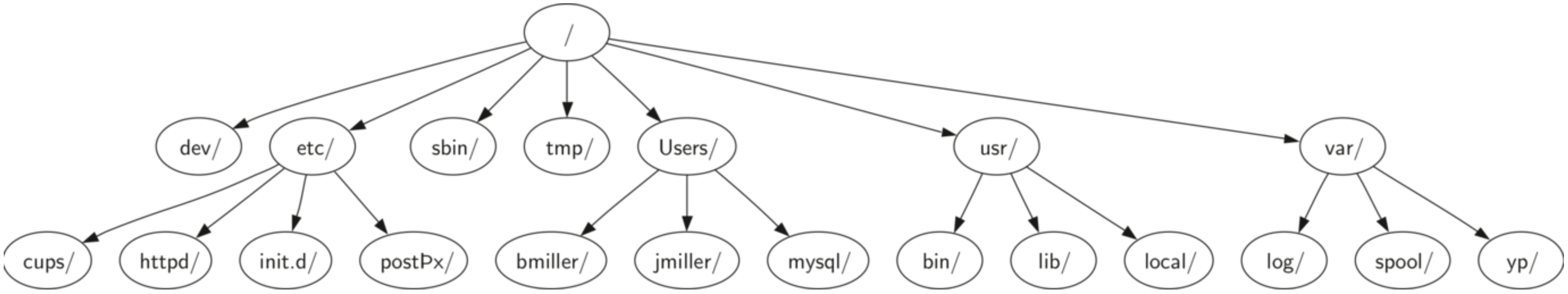
If language is a tree, it's a gnarly and complex one: it's not as if languages subdivide and never re-merge. Take English: at various points in its history (usually thanks to invading or being invaded) it's come into contact with Latin (the Romans), Germanic languages (Vikings), French (the Normans) and others. It has constantly evolved and assimilated words and structures, and every few hundred years changes so much that a speaker of a few hundred years before would struggle to understand.

Perhaps language is less of a tree than a river: while the banks stay roughly where they are, the water is always moving, separating, recombining … but never the same.

This language tree is divided into two parts: the ***Centum*** *(🟦) languages are western European, and the* ***Satem*** *(🟨) languages are eastern European and Asian. Tocharian is an exception, but it's complicated. Languages marked with an asterisk (*) are official languages of the European Union.*

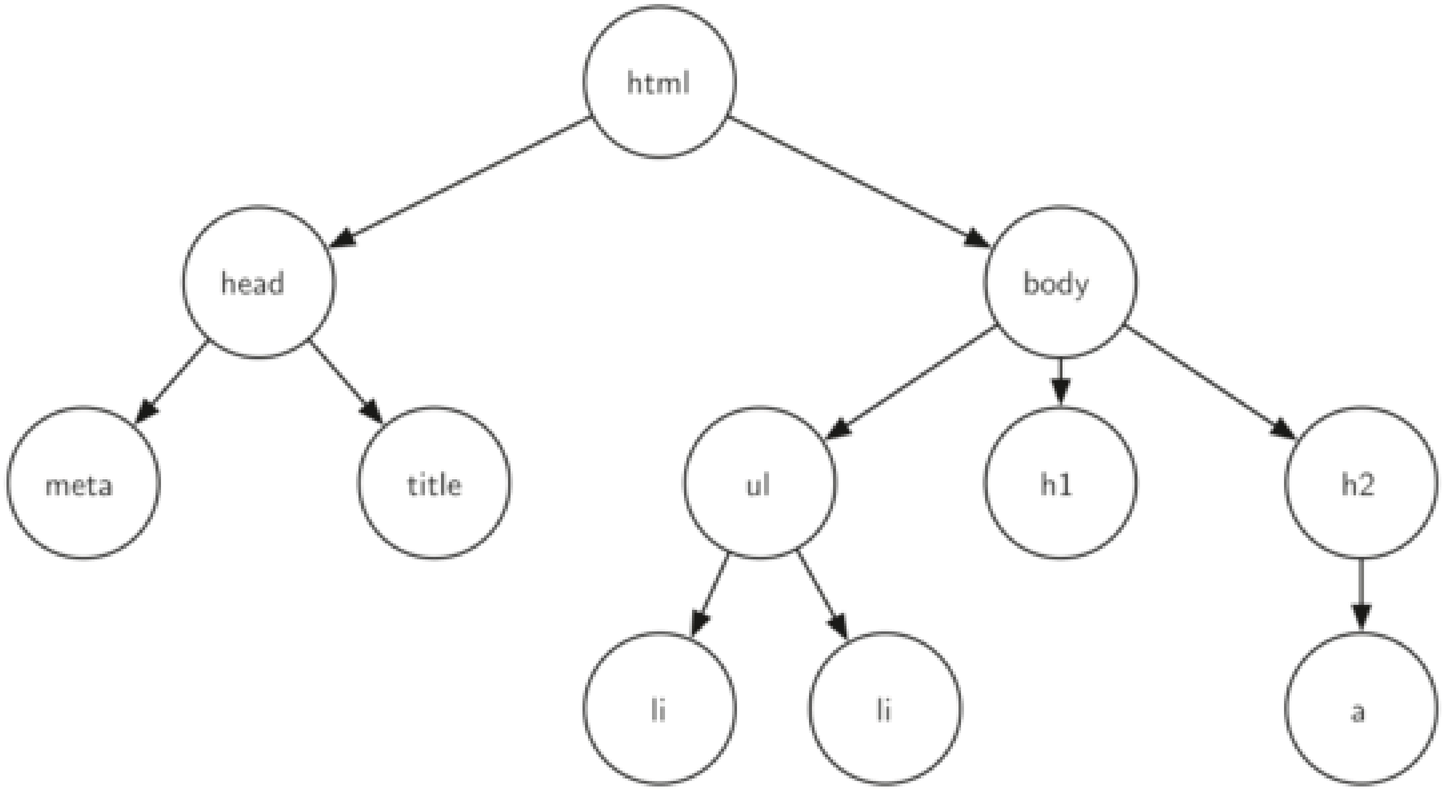
GETTING CIVILISED
066 – 067

To organize file-systems

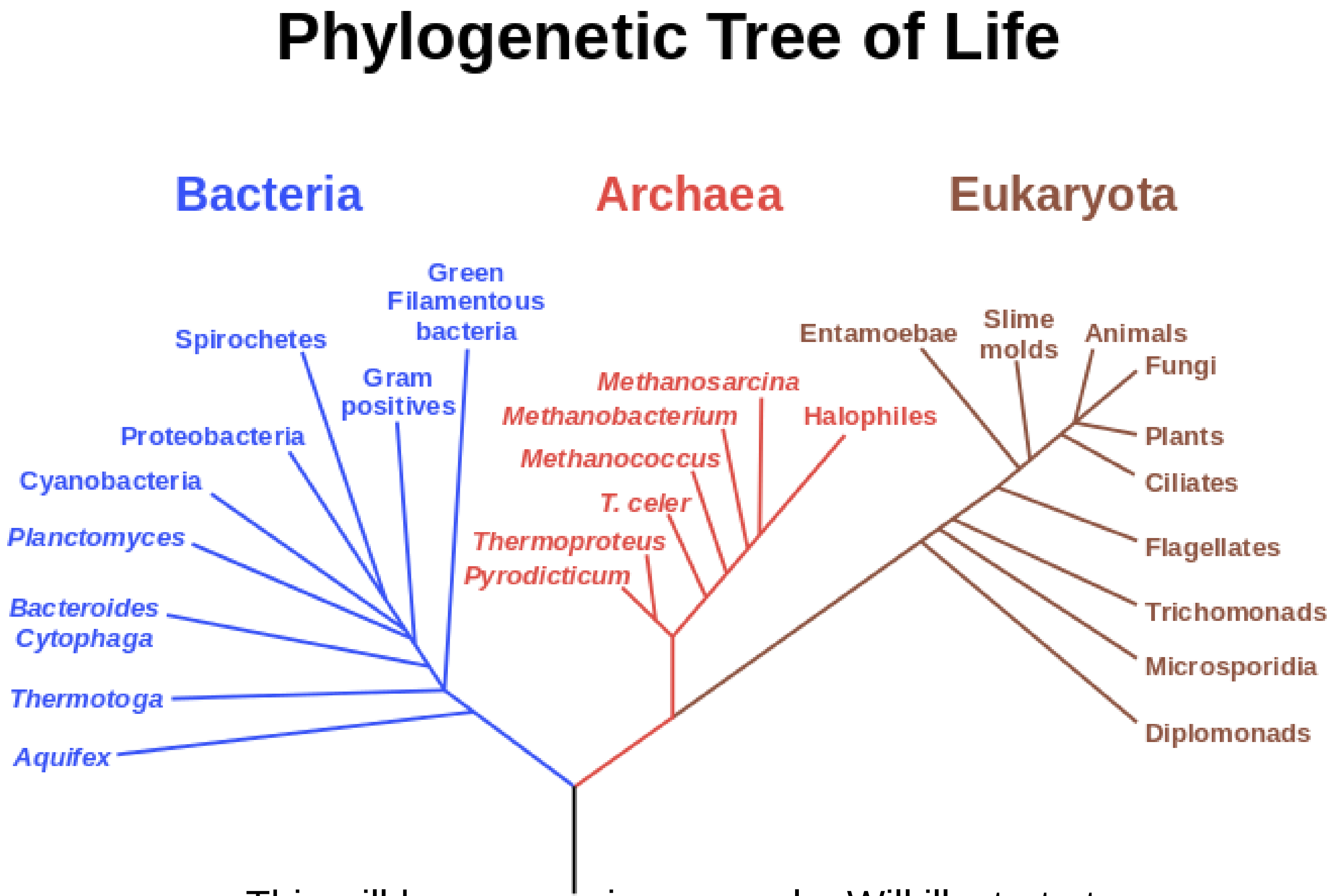


Unix file system

Markup elements in a webpage



To store phylogenetic data



This will be our running example. Will illustrate tree concepts using actual phylogenetic data.

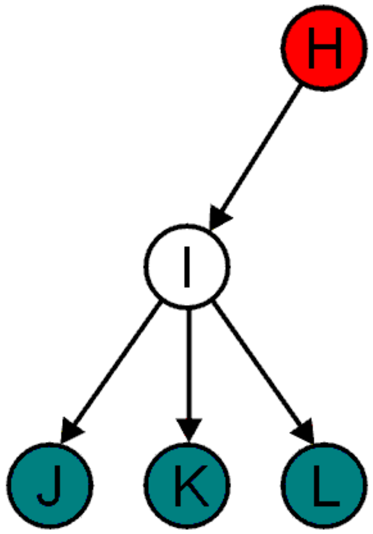
Terminology

All nodes will have zero or more child nodes or *children*

- I has three children: J, K and L

For all nodes other than the root node, there is one parent node

- H is the parent of I

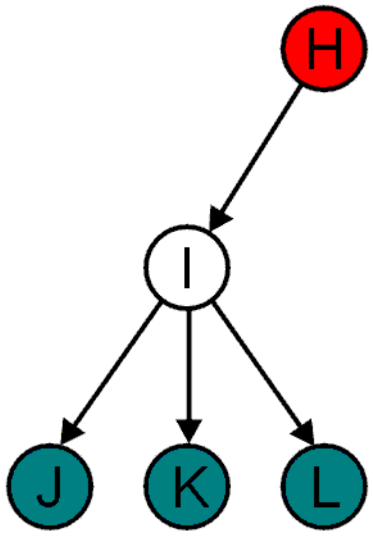


Terminology

The *degree* of a node is defined as the number of its children: deg(I) = 3

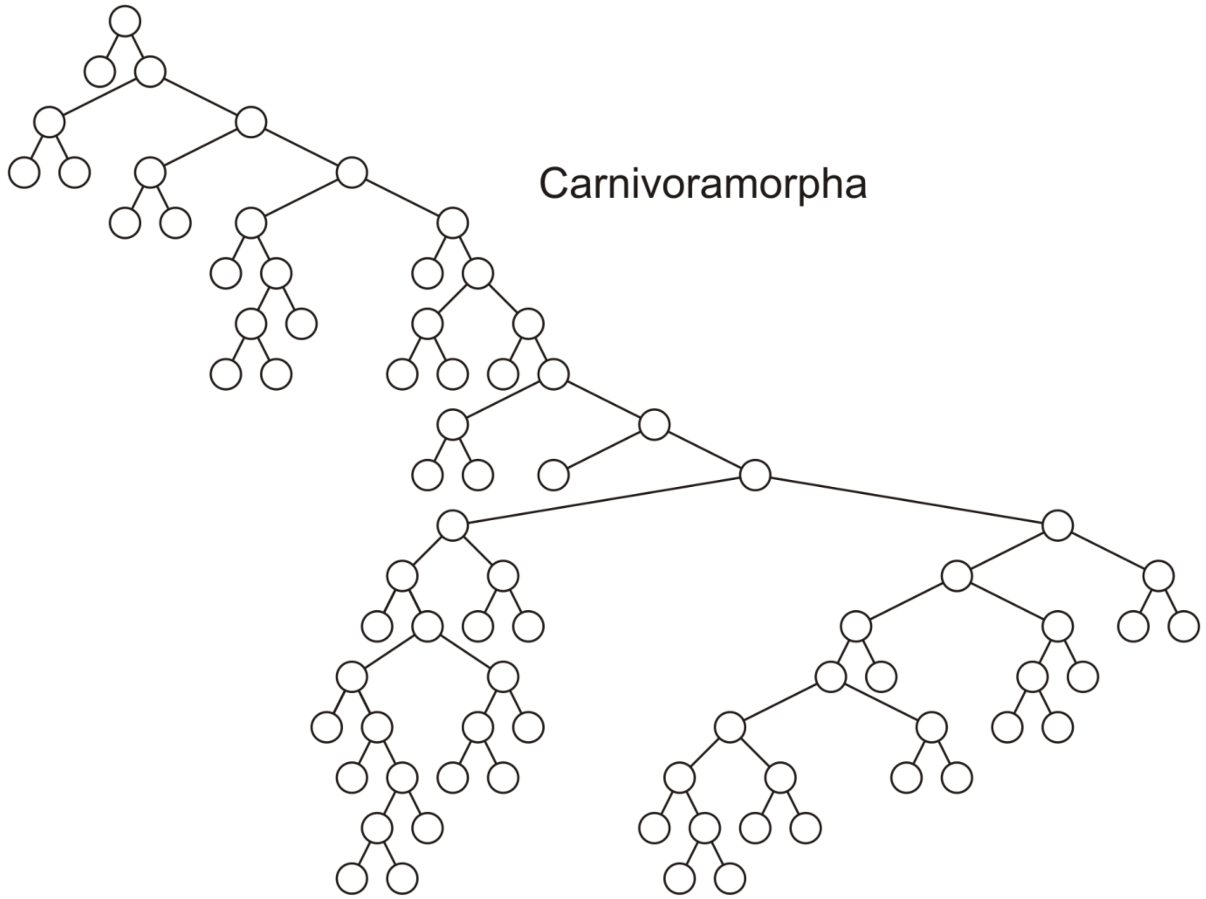
Nodes with the same parent are *siblings*

– J, K, and L are siblings



Terminology

Phylogenetic trees have nodes with degree 2 or 0:

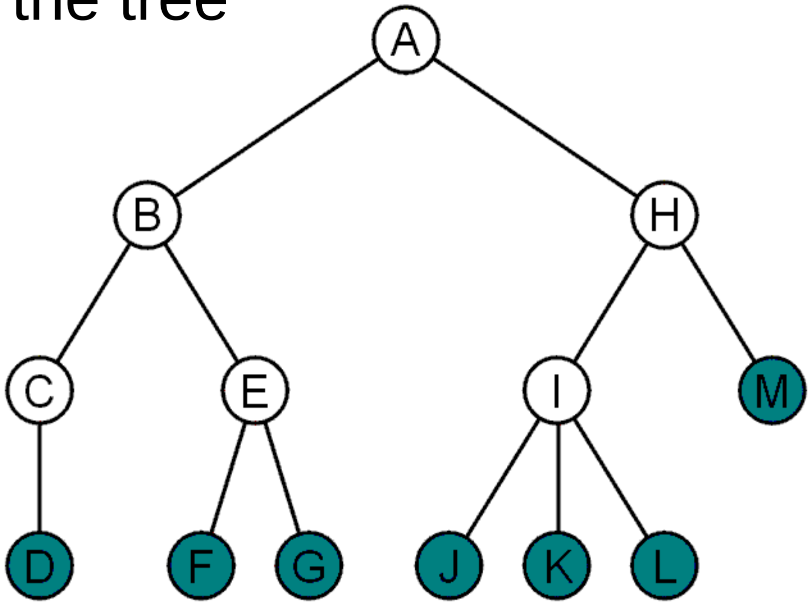


Wesley-Hunt, G. D.; Flynn, J. J. "Phylogeny of the Carnivora: basal relationships among the Carnivoramorphans, and assessment of the position of 'Miacoidea'

Terminology

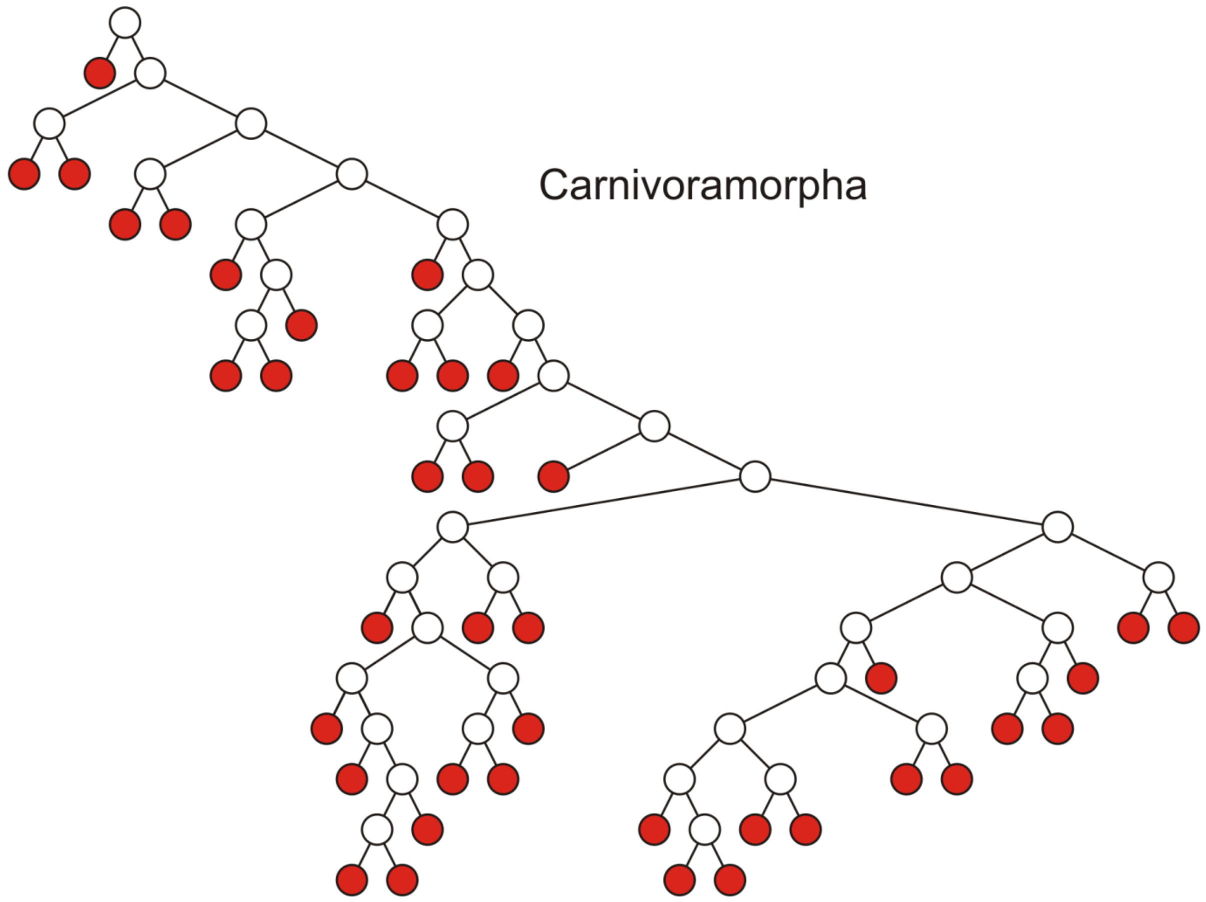
Nodes with degree zero are also called *leaf nodes*

All other nodes are said to be *internal nodes*, that is, they are internal to the tree



Terminology

Leaf nodes:



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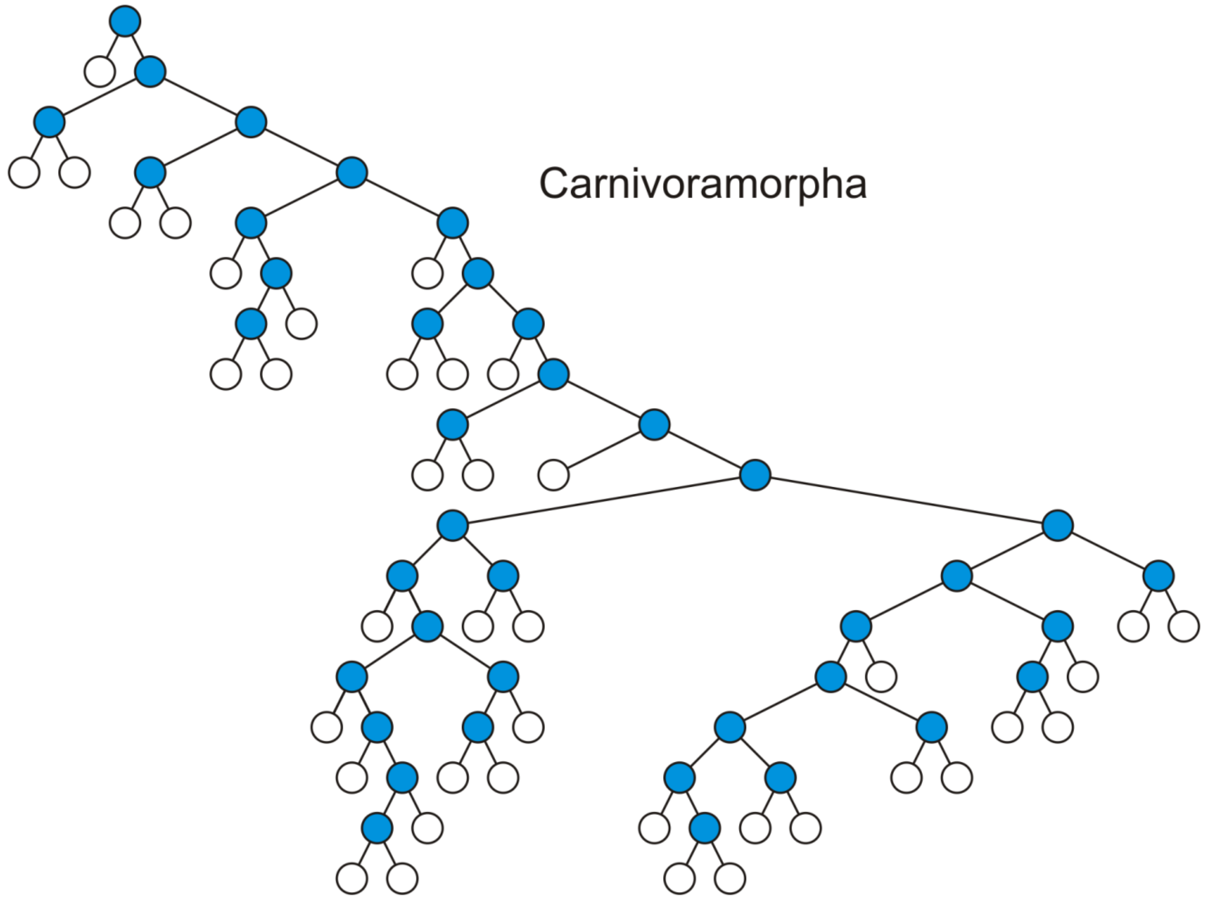
Terminology

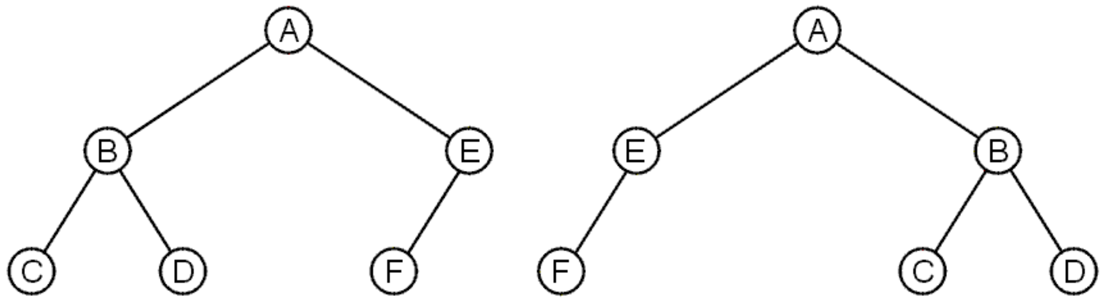
Internal nodes:



Wesley-Hunt, G. D.; Flynn, J. J. "Phylogeny of the Carnivora: basal relationships among the Carnivoramorphans, and assessment of the position of 'Miacoidea'

Terminology

These trees are equal if the order of the children is ignored (*Unordered trees*)

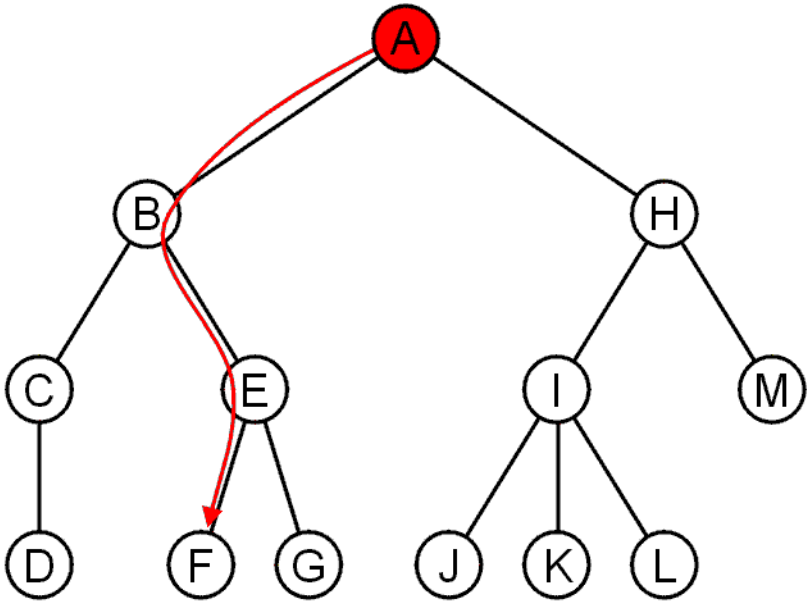


They are different if order is relevant (*ordered trees*)

- We will usually examine ordered trees (linear orders)
- In a hierarchical ordering, order is not relevant

Terminology

The shape of a rooted tree gives a natural flow from the *root node*, or just *root*



Terminology

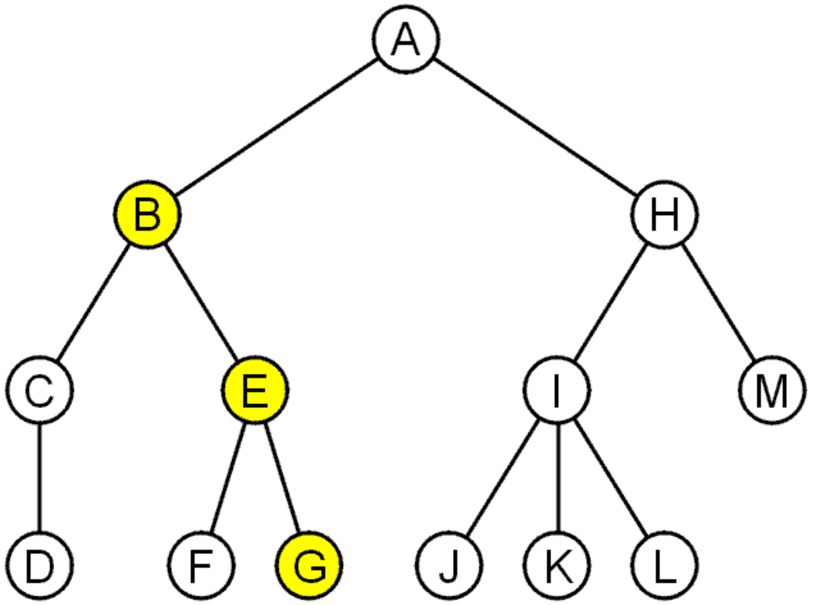
A path is a sequence of nodes

$(a_0, a_1, ..., a_n)$

where a_{k+1} is a child of a_k is

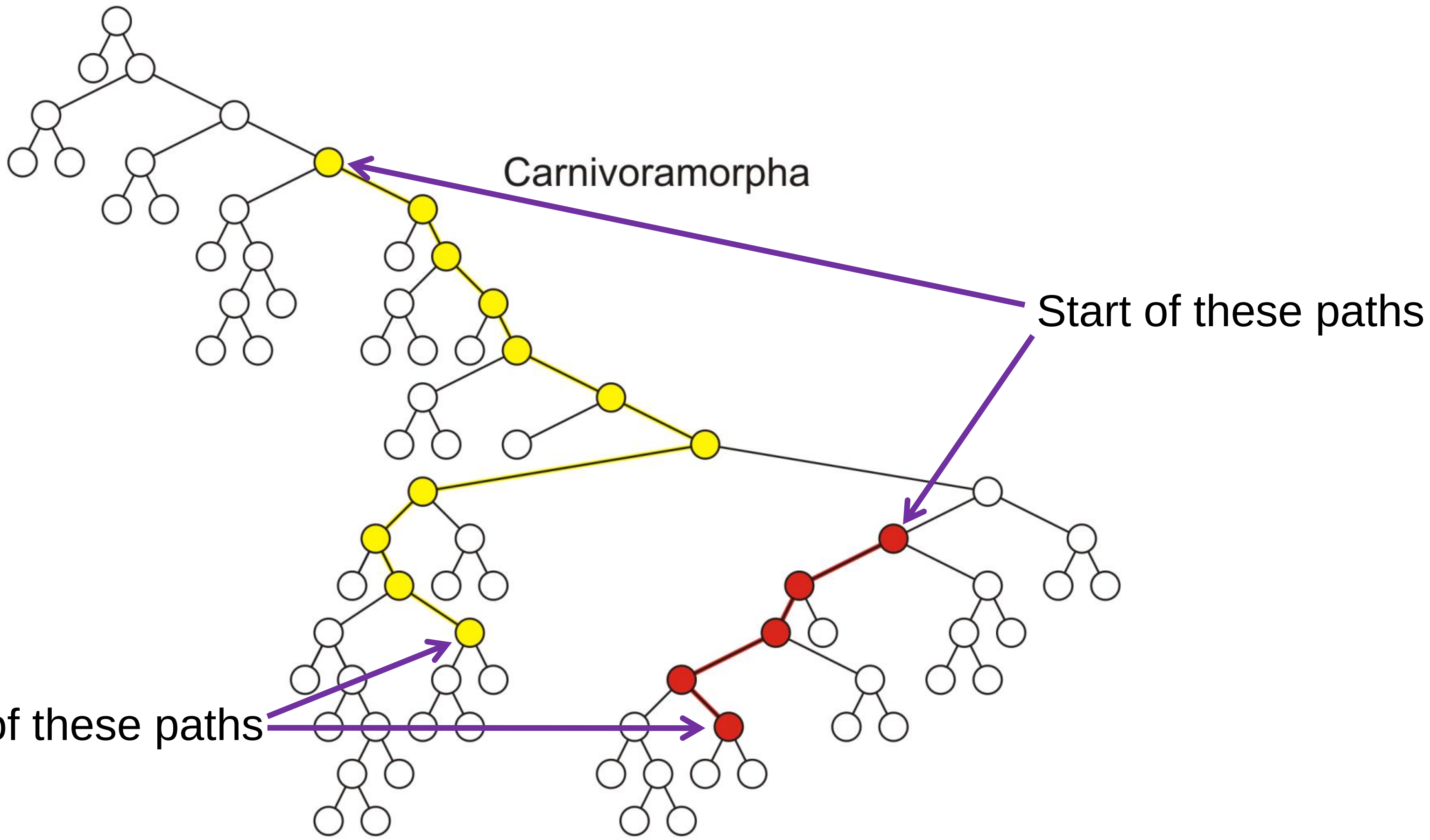
The length of this path is n

E.*g*., the path (B, E, G)
has length 2



Terminology

Paths of length 10 (11 nodes) and 4 (5 nodes)



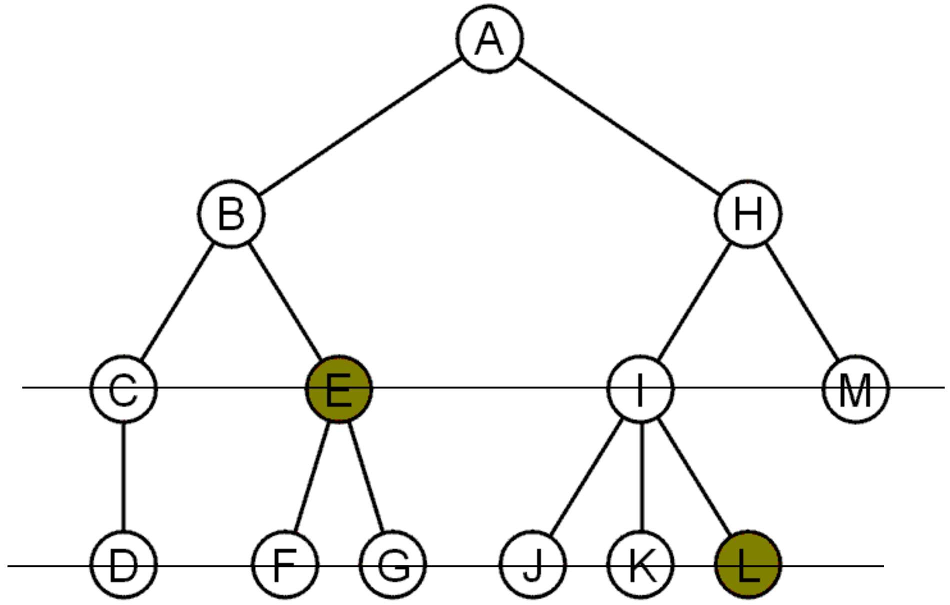
Wesley-Hunt, G. D.; Flynn, J. J. "Phylogeny of the Carnivora: basal relationships among the Carnivoramorphans, and assessment of the position of 'Miacoidea'

Terminology

For each node in a tree, there exists a unique path from the root node to that node

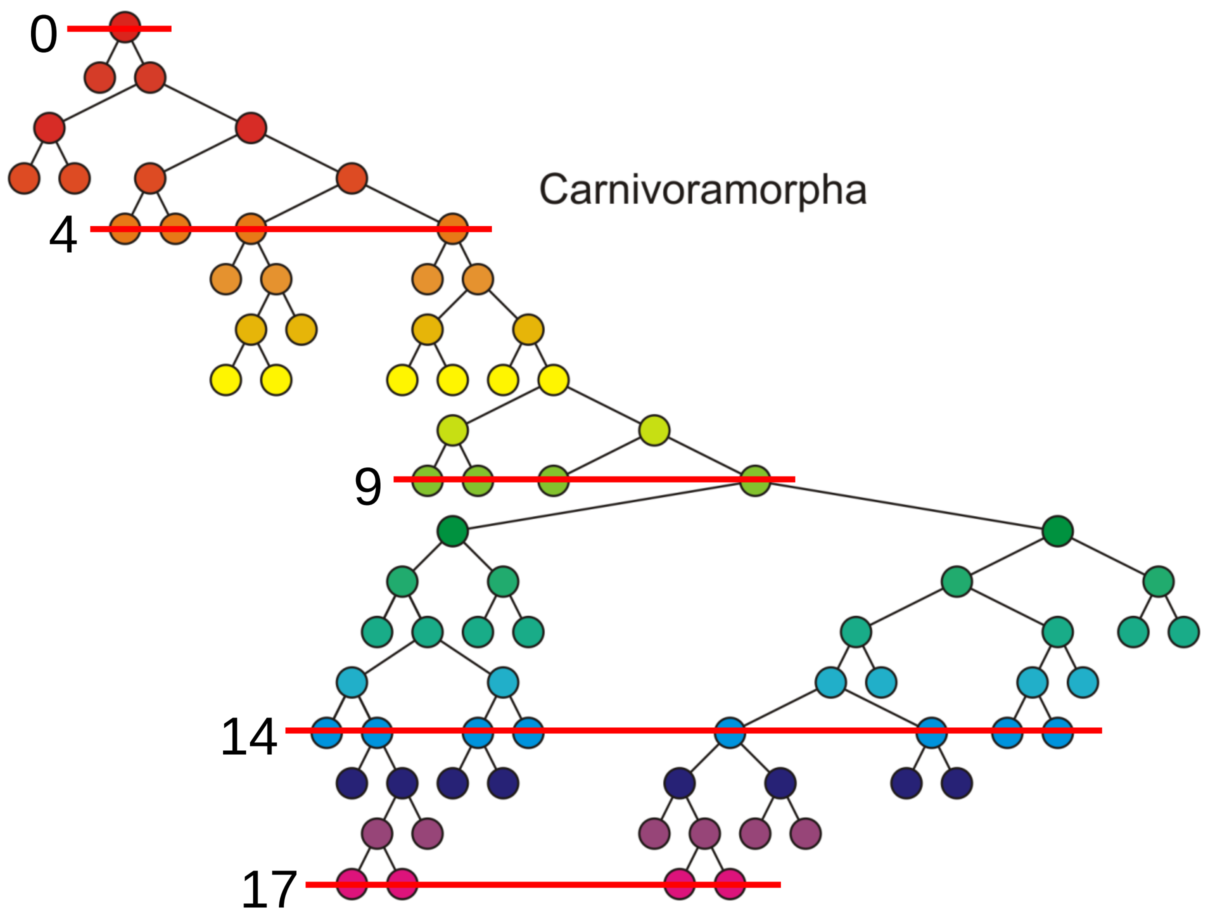
The length of this path is the *depth* of the node, *e*.*g*.,

- E has depth 2
- L has depth 3



Terminology

Nodes of depth up to 17



Wesley-Hunt, G. D.; Flynn, J. J. "Phylogeny of the Carnivora: basal relationships among the Carnivoramorphans, and assessment of the position of 'Miacoidea'

Terminology

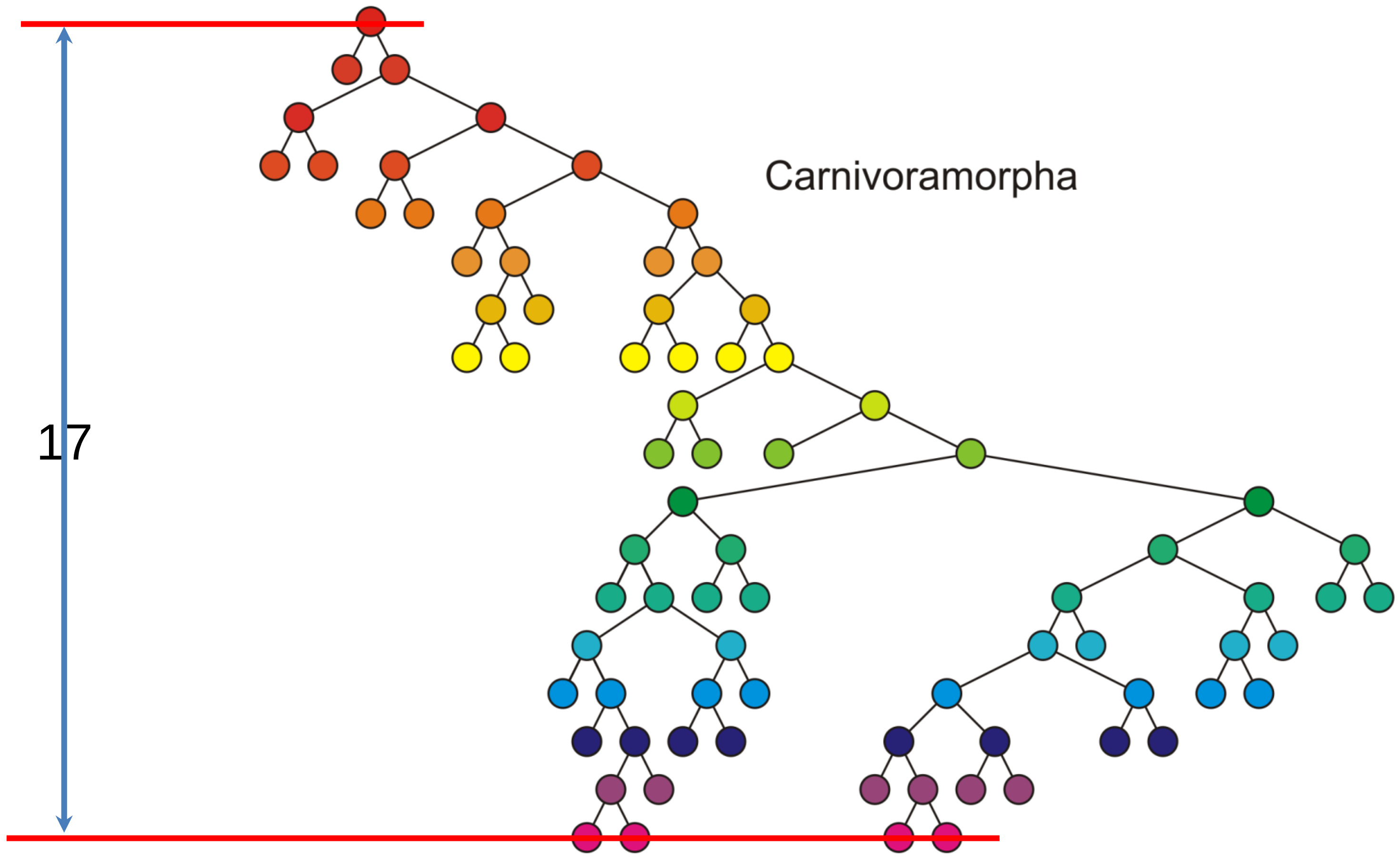
The *height* of a tree is defined as the maximum depth of any node within the tree

The height of a tree with one node is 0
- Just the root node

For convenience, we define the height of the empty tree to be -1

Terminology

The height of this tree is 17



Wesley-Hunt, G. D.; Flynn, J. J. “Phylogeny of the Carnivora: basal relationships among the Carnivoramorphans, and assessment of the position of ‘Miacoidea’

Terminology

If a path exists from node a to node b:

- a is an *ancestor* of b
- b is a *descendent* of a

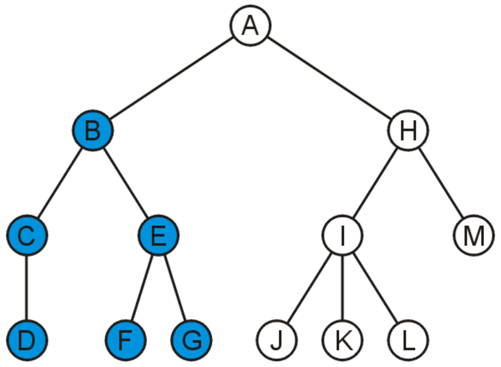
Thus, a node is both an ancestor and a descendant of itself

- We can add the adjective *strict* to exclude equality: a is a *strict descendent* of b if a is a descendant of b but $a \neq b$

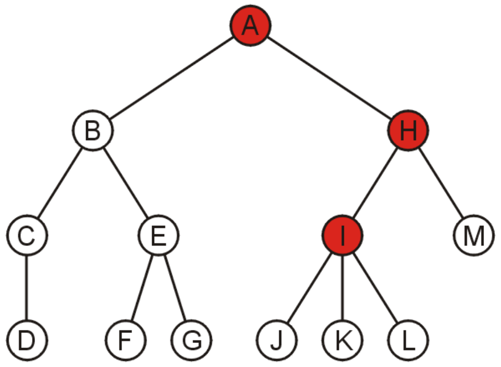
The root node is an ancestor of all nodes

Terminology

The descendants of node B are B, C, D, E, F, and G:

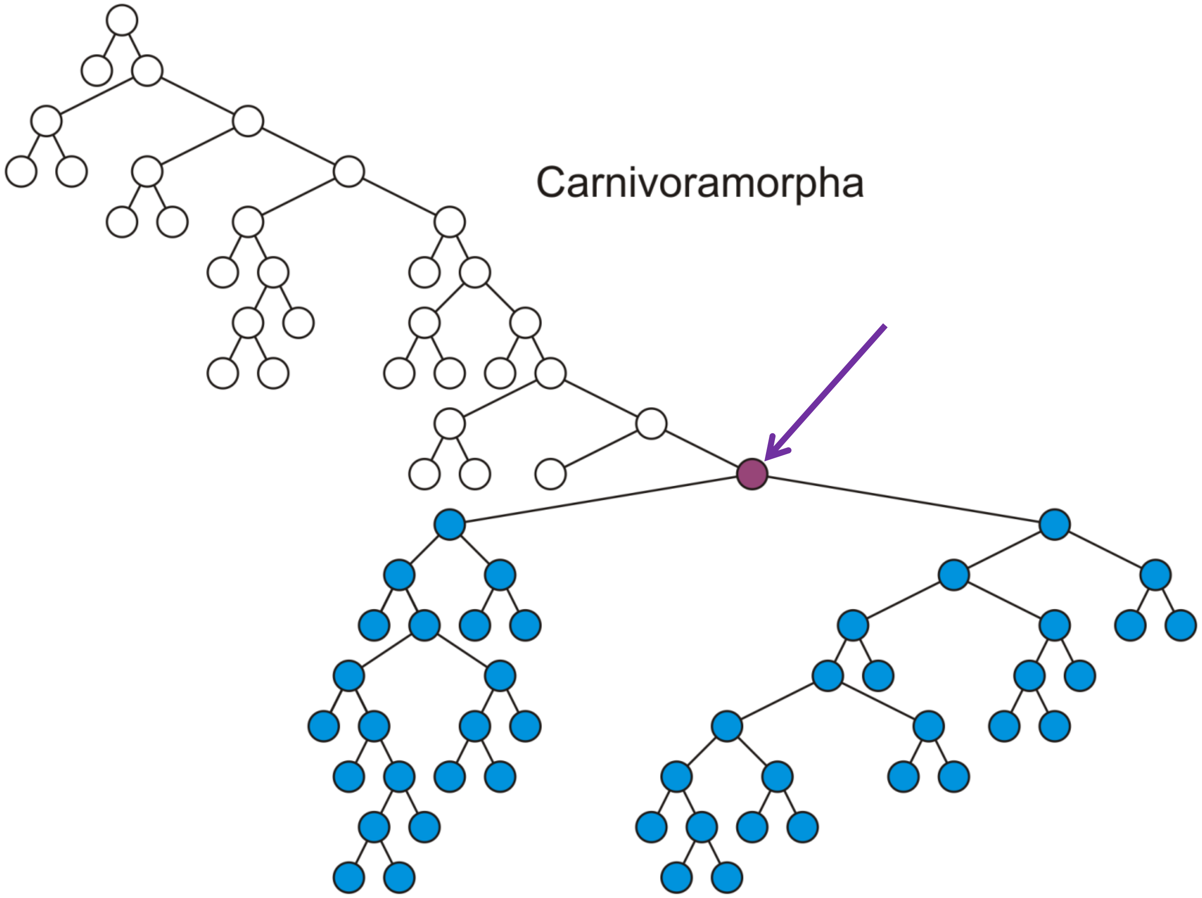


The ancestors of node I are I, H, and A:



Terminology

All descendants (including itself) of the indicated node



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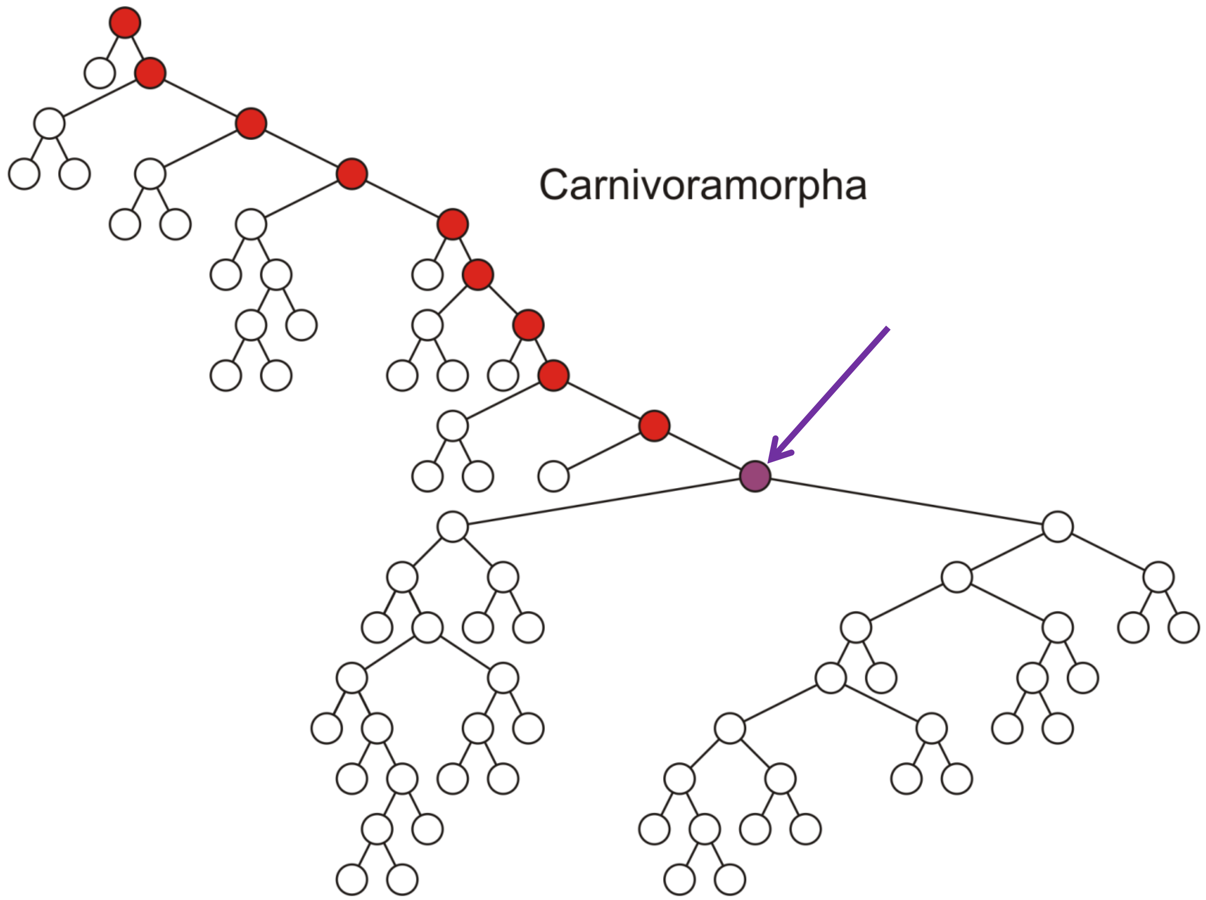

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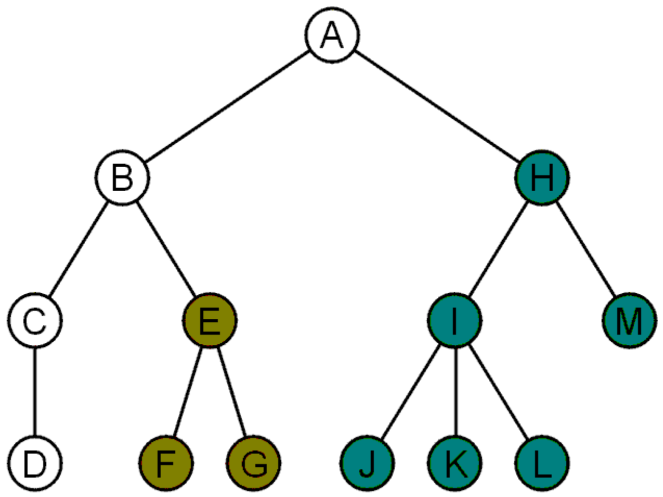
Terminology

Another approach to a tree is to define the tree recursively:

- A degree-0 node is a tree
- A node with degree n is a tree if it has n children and all of its children are disjoint trees (*i.e.*, with no intersecting nodes)

Given any node a within a tree with root r, the collection of a and all of its descendants is said to be a *subtree of the tree with root* a



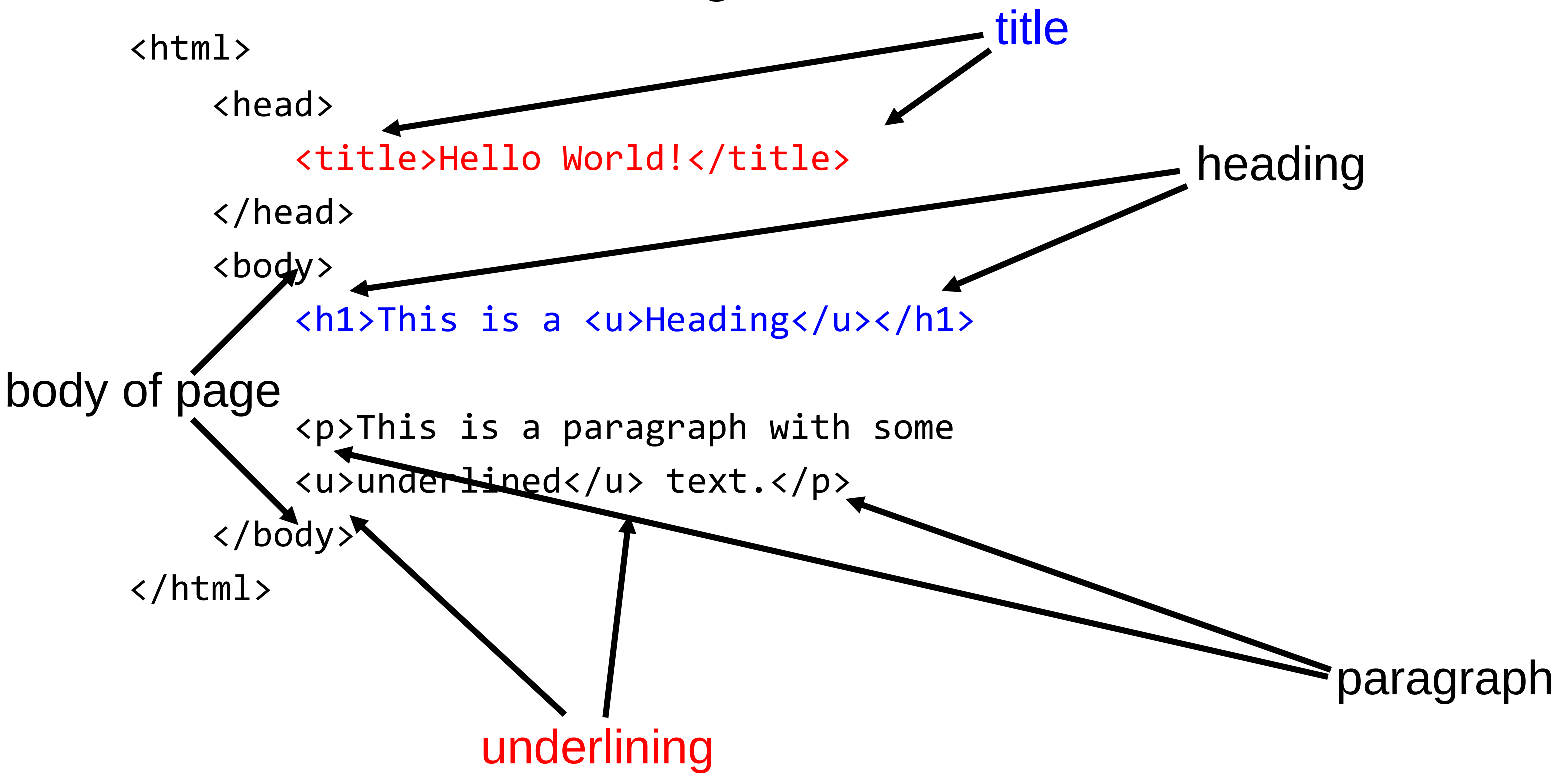
Example: XHTML

Consider the following XHTML document

```
<html>
    <head>
        <title>Hello World!</title>
    </head>
    <body>
        <h1>This is a <u>Heading</u></h1>

        <p>This is a paragraph with some
        <u>underlined</u> text.</p>
    </body>
</html>
```

Example: XHTML

Consider the following XHTML document

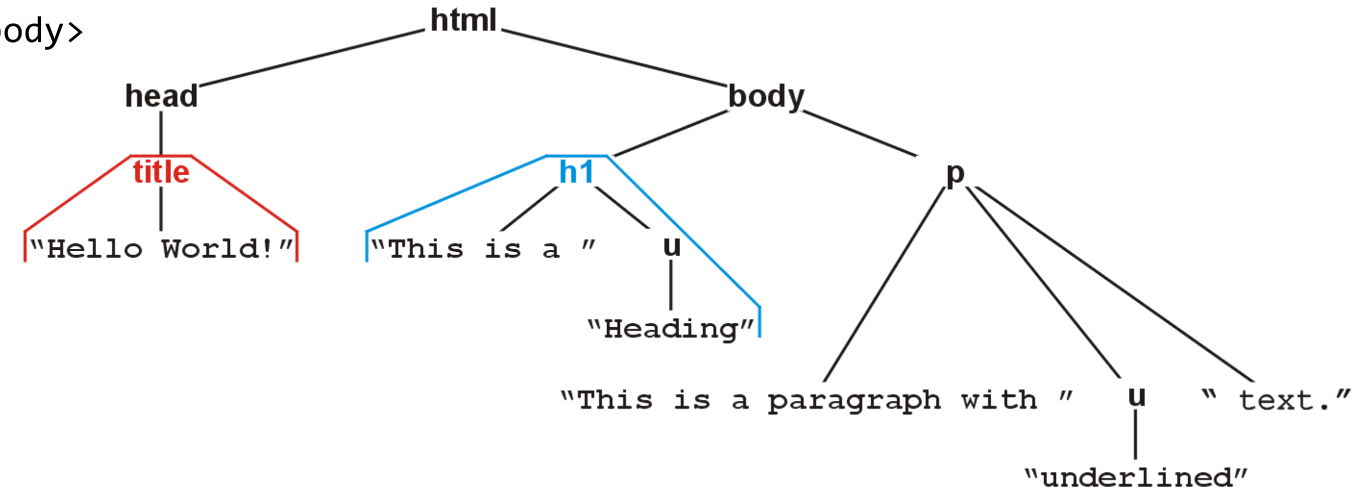


```
<html>
    <head>
        <title>Hello World!</title>
    </head>
    <body>
        <h1>This is a <u>Heading</u></h1>

        <p>This is a paragraph with some
        <u>underlined</u> text.</p>
    </body>
</html>
```

Example: XHTML

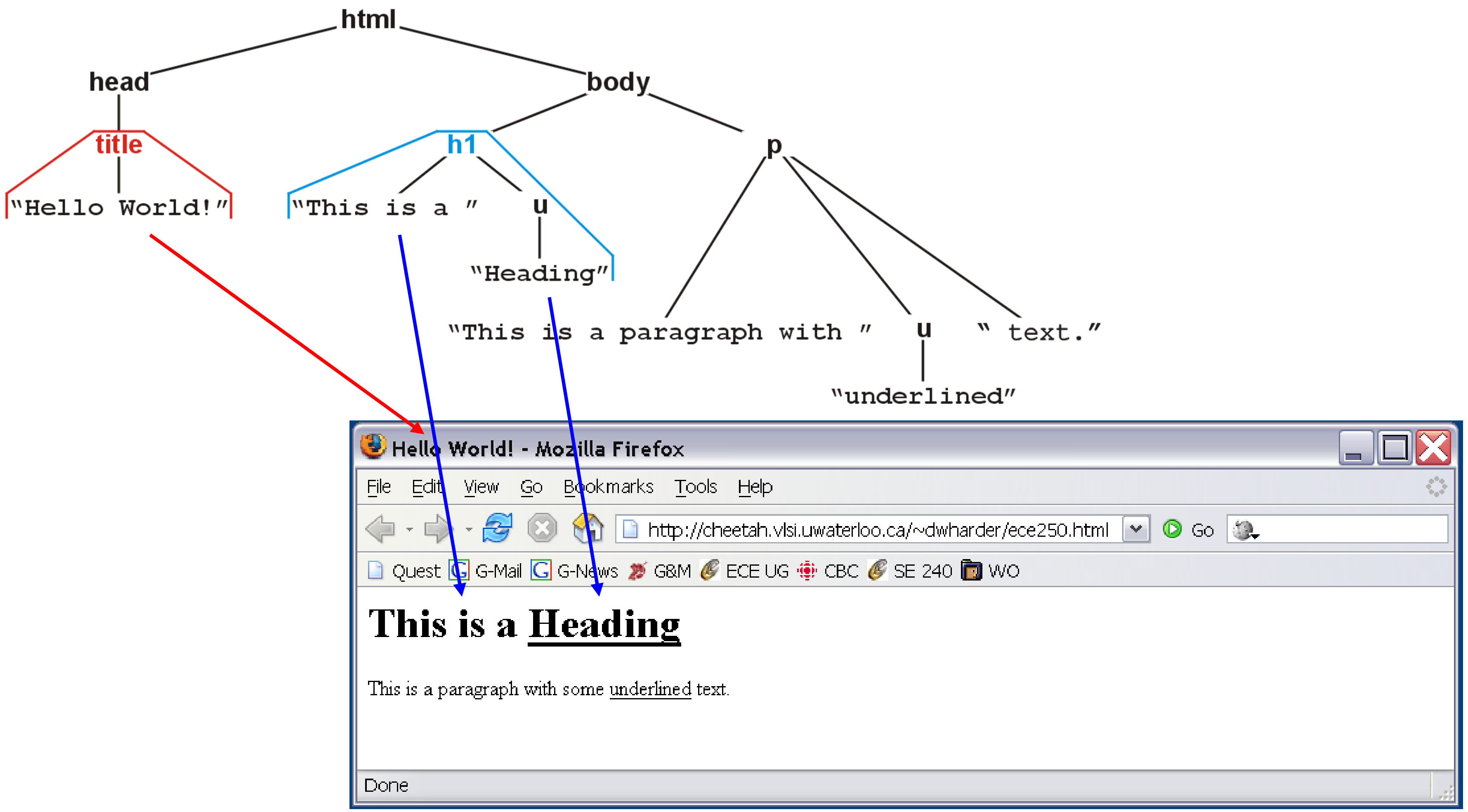
The nested tags define a tree rooted at the HTML tag

```
<html>
    <head>
        <title>Hello World!</title>
    </head>
    <body>
        <h1>This is a <u>Heading</u></h1>

        <p>This is a paragraph with some
        <u>underlined</u> text.</p>
    </body>
</html>
```



Example: XHTML

Web browsers render this tree as a web page


html
head
body
title
h1
p
"Hello World!"
"This is a "
u
"Heading"
"This is a paragraph with "
u
" text."
"underlined"
Hello World! - Mozilla Firefox
File Edit View Go Bookmarks Tools Help
http://cheetah.vlsi.uwaterloo.ca/~dwharder/ece250.html
Go
Quest G-Mail G-News G&M ECE UG CBC SE 240 WO
Done

This is a <u>Heading</u>

This is a paragraph with some <u>underlined</u> text.

Iterator ADT

Most ADTs in Java can provide an iterator object, used to traverse all the data in any linear ADT.

Iterator Interface

```
public interface Iterator<E>{

    boolean hasNext();

    E next();

   void remove(); // Optional

 }
```

Getting an Iterator

You get an iterator from an ADT by calling the method `iterator();`

```
Iterator<Integer> iter = myList.iterator();
```

Now a simple while loop can process each data value in the ADT:

```
while(iter.hasNext()) {
    process iter.next()
}
```

Adding Iterators to SimpleArrayList is easy

First, we add the `iterator()` method to `SimpleArrayList`:

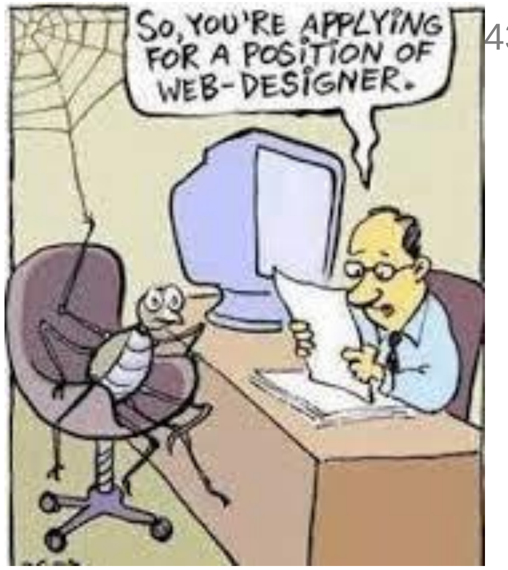
```
public Iterator<E> iterator(){
    return new
      ArrayListIterator<E>(this);
}
```

Then we implement the iterator class for Lists:

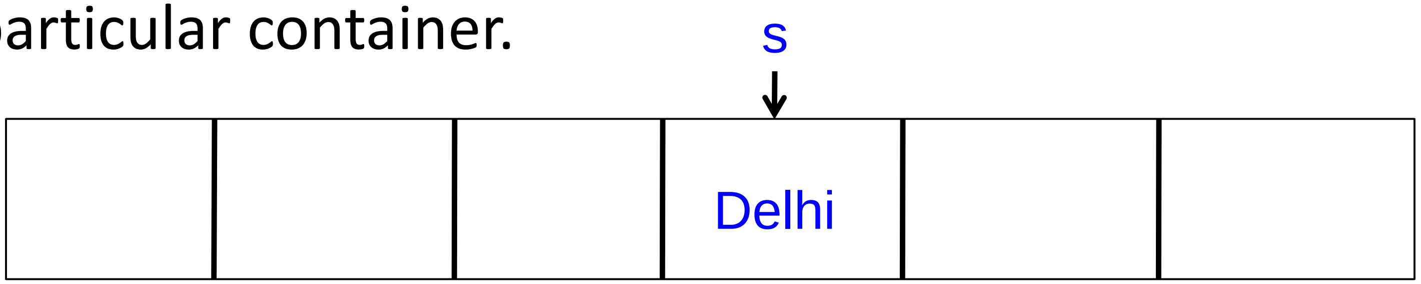
```
import java.util.*;
public class ArrayListIterator<E>
 implements Iterator<E> {
    // *** fields ***
    private SimpleArrayList<E> list;
    private int curPos;

    public ArrayListIterator(
          SimpleArrayList<E> list) {
        this.list = list;
        curPos = 0;
    }
```

```
  public boolean hasNext() {
      return curPos < list.size();
   }

  public E next() {
     if (!hasNext()) throw
        new NoSuchElementException();

     E result = list.get(curPos);
     curPos++;
     return result;
  }

   public void remove() {
       throw new UnsupportedOperationException();
   }

}
```

Position ADT



- Say vector contains an element "Delhi" that we want to keep track of
- The index of the element may keep changing depending on insert/delete operations
- A position *s*, may be associated with the element *Delhi (*say at time of insertion)
- s lets us access *Delhi* via s.element() even if the index of *Delhi* changes in the container, unless we explicitly remove *s*
- A position ADT is associated with a particular container.



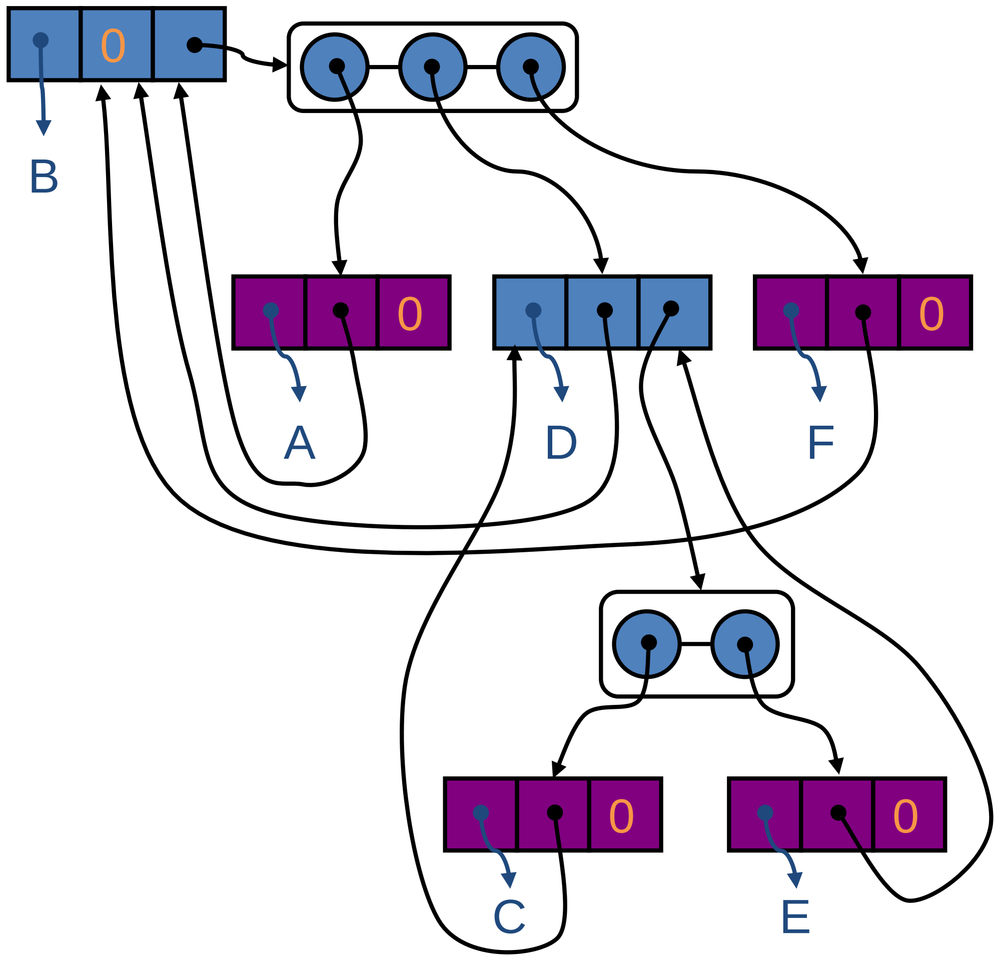
Tree ADT

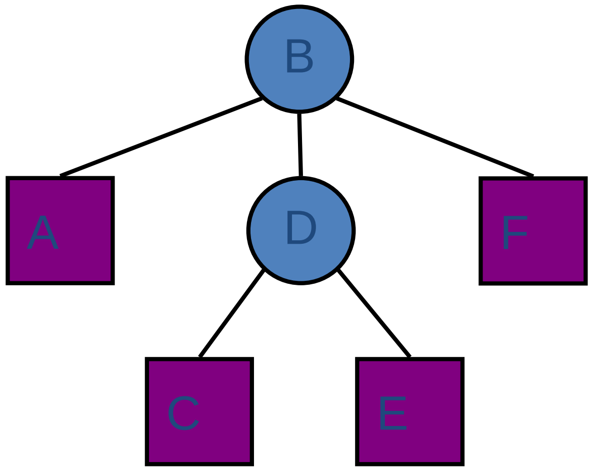
- We use positions to abstract nodes
- Generic methods:
  - integer size()
  - boolean isEmpty()
  - objectIterator elements()
  - positionIterator positions()
- Accessor methods:
  - position root()
  - position parent(p)
  - positionIterator children(p)

- Query methods:
  - boolean isInternal(p)
  - boolean isLeaf (p)
  - boolean isRoot(p)
- Update methods:
  - swapElements(p, q)
  - object replaceElement(p, o)
- Additional update methods may be defined by data structures implementing the Tree ADT

A Linked Structure for General Trees

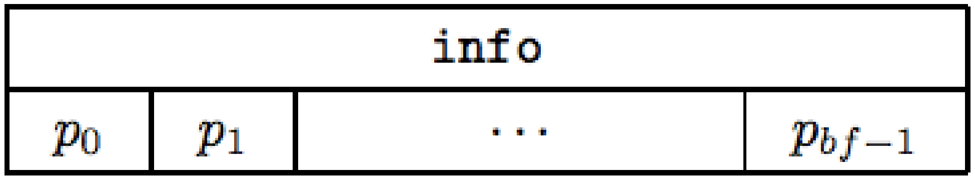
- A node is represented by an object storing
  - Element
  - Parent node
  - Sequence of children nodes
- Node objects implement the Position ADT





Tree using Array

- Each node contains a field for data and an array of pointers to the children for that node
  - – Missing child will have null pointer
- Tree is represented by pointer to root
- Allows access to i^{th} child in O(1) time
- Very wasteful in space when only few nodes in tree have many children (most pointers are null)



info			
p_0	p_1	$\cdots$	p_{bf-1}

Tree Traversals

- A *traversal* visits the nodes of a tree in a systematic manner

- We will see three types of traversals
  - Pre-order
  - Post-order
  - In-order

Flavors of (Depth First) Traversal

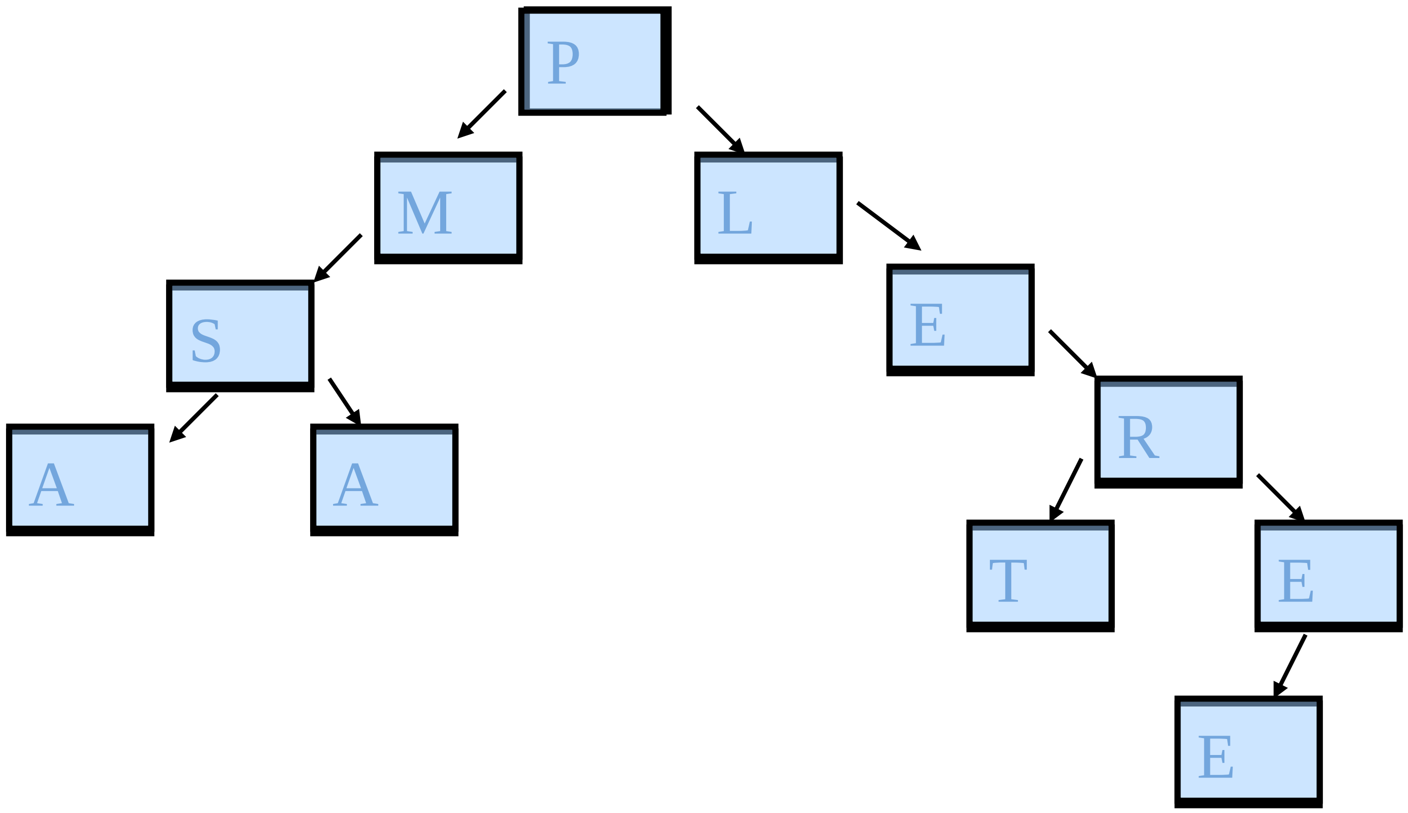
- In a *preorder traversal*, a node is visited before its descendants
- In a *postorder traversal, a node is visited after its* descendants
- In an *inorder traversal* a node is visited after its left subtree and before its right subtree

Preorder Traversal

- Process the root
- Process the nodes in the all subtrees in their order

```
Algorithm preOrder(v)
    visit(v)
    for each child w of v
        preOrder(w)
```

Preorder Traversal



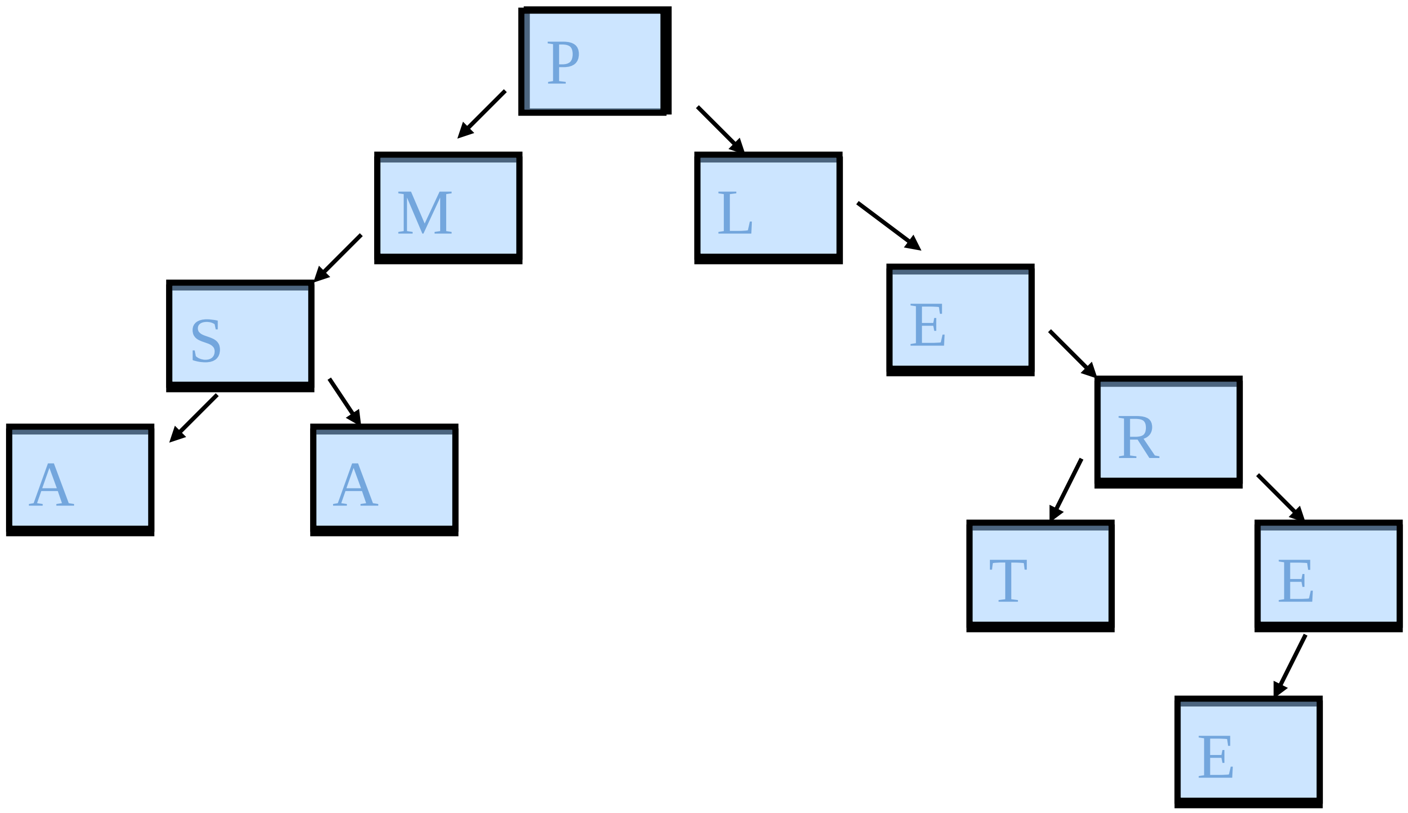
Preorder traversal: node is visited before its descendants

Postorder traversal

1. Process the nodes in all subtrees in their order
2. Process the root

```
Algorithm postOrder(v)
    for each child w of v
        postOrder(w)
    visit(v)
```

Postorder Traversal



Postorder traversal: node is visited before its descendants

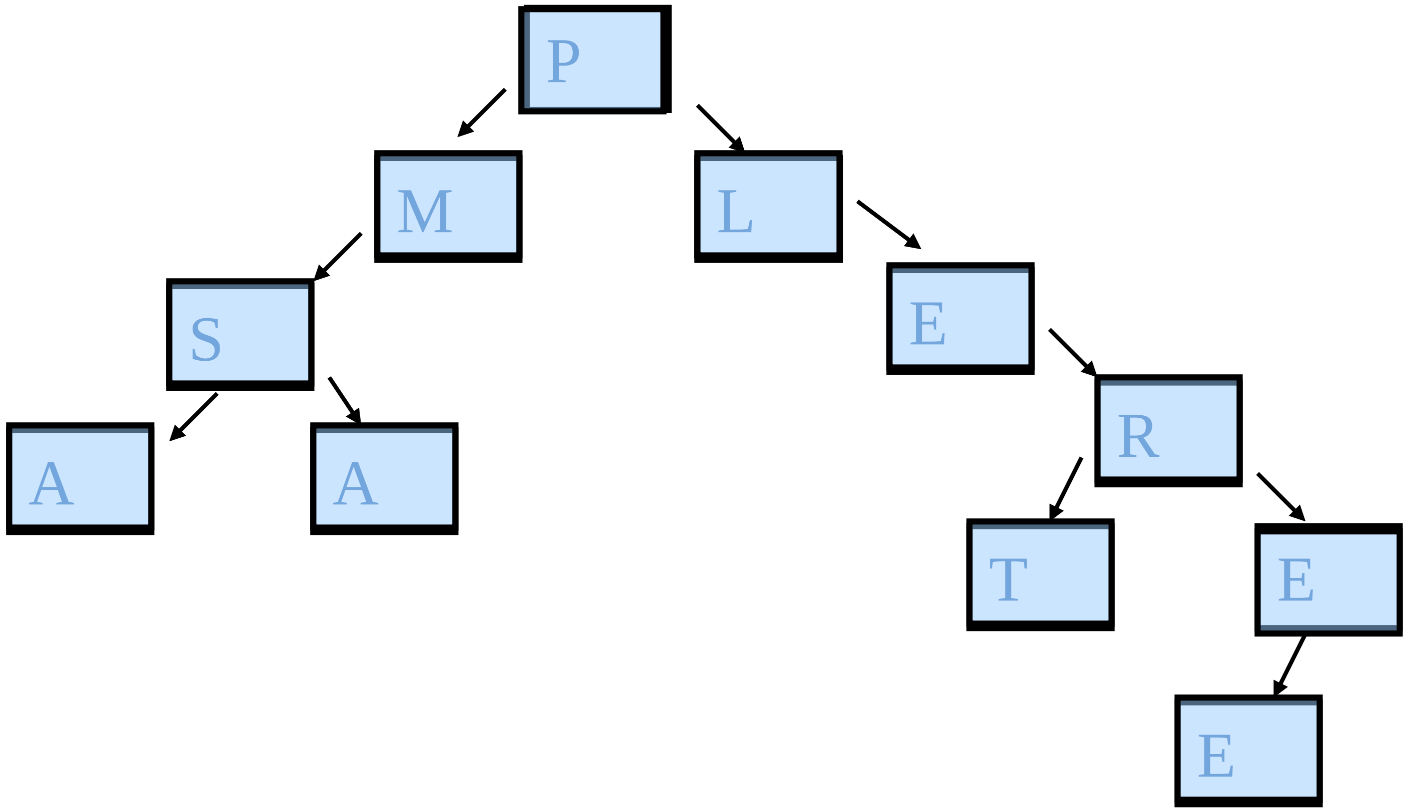
Inorder traversal

1. Process the nodes in the left subtree
2. Process the root
3. Process the nodes in the right subtree

```
Algorithm InOrder(v)
    InOrder(v->left)
    visit(v)
    InOrder(v->right)
```

For simplicity, we consider tree having at most 2 children, though it can be generalized.

Inorder Traversal



Inorder traversal: node is visited after its left subtree

and before its right subtree

Non-Recursive preorder traversal

1. Start from root.
2. Print the node.
3. Push right child onto to stack.
4. Push left child onto to stack.
5. Pop node from the stack.
6. Repeat Step 2 to 5 till stack is not empty.

Computing Height of Tree

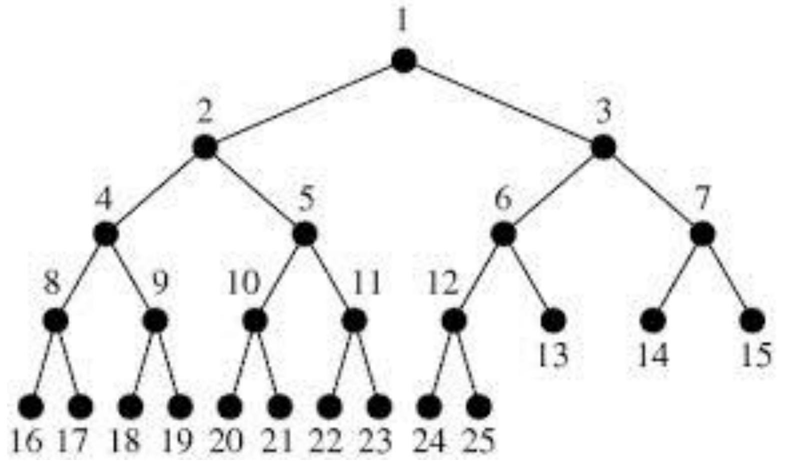
Can be computed using the following idea:

1. The height of a leaf node is 0
2. The height of a node other than the leaf is the maximum of the height of the left subtree and the height of the right subtree plus 1.

Height(v) = max[height(v→left) + height(v→right)] + 1

Details left as exercise.

Binary Trees

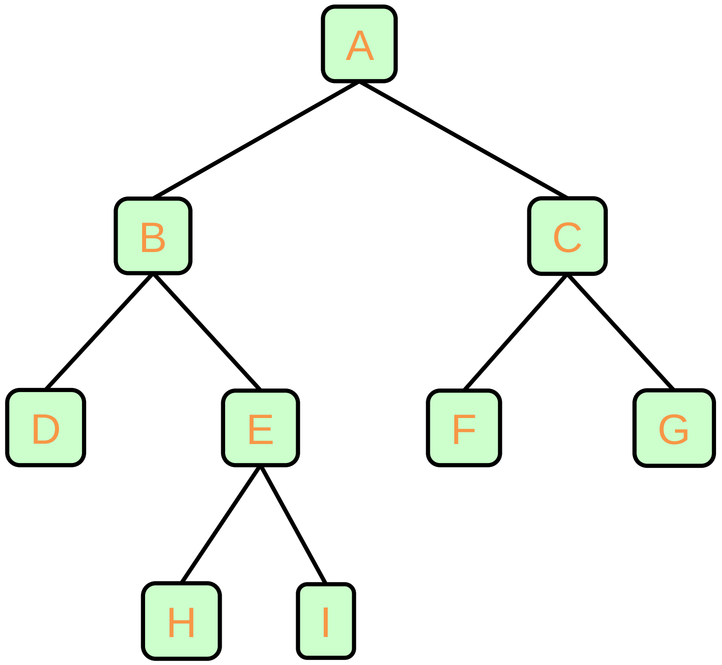


Every node has degree up to 2.
Proper binary tree: each internal node has degree exactly 2.

Binary Tree

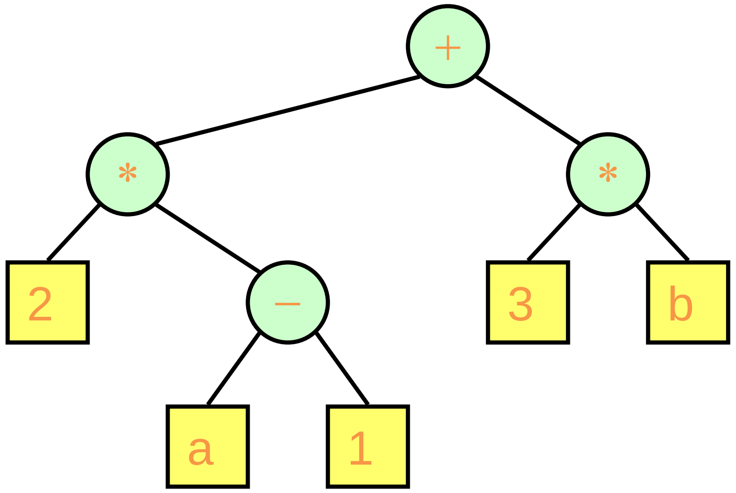
- A *binary tree* is a tree with the following properties:
  - Each internal node has two children
  - The children of a node are an ordered pair
- We call the children of an internal node *left child* and *right child*
- Alternative recursive definition: a binary tree is either
  - a tree consisting of a single node, or
  - a tree whose root has an ordered pair of children, each of which is a disjoint binary tree

- Applications:
  - arithmetic expressions
  - decision processes
  - searching

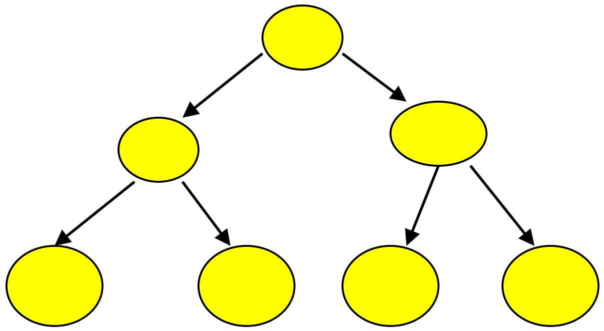


Arithmetic Expression Tree

- Binary tree associated with an arithmetic expression
  - internal nodes: operators
  - leaves: operands
- Example: arithmetic expression tree for the expression $(2 * (a - 1) + (3 * b))$



How many leaves L does a proper binary tree of heigh h have?



The number of leaves at depth d = 2^d

If the height of the tree is h it has 2^h leaves.

L = $2^{h.}$

What is the height h of a proper binary tree with L leaves?

leaves = 1	height = 0
leaves = 2	height = 1
leaves = 4	height = 2
leaves = L	height = $\text{Log}_2 L$

Since $L = 2^h$

$\log_2 L = \log_2 2^h$

$h = \log_2 L$

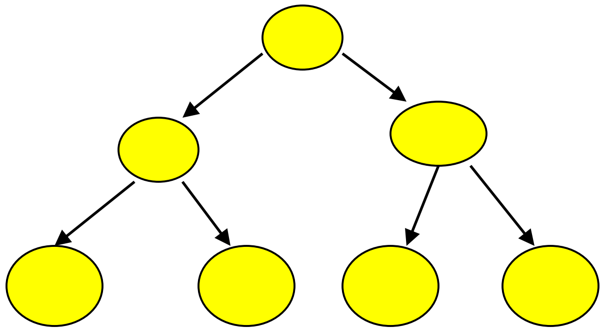
The number of internal nodes of a proper binary tree of height *h* is ?

Internal nodes = 0	height = 0
Internal nodes = 1	height = 1
Internal nodes = 1 + 2	height = 2
Internal nodes = 1 + 2 + 4	height = 3

$1 + 2 + 2^2 + . . . + 2^{h-1} = 2^h - 1$ Geometric series

Thus, a complete binary tree of height = h has 2^h-1 internal nodes.

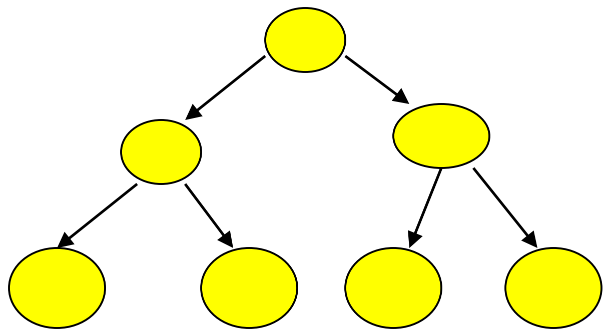
The number of nodes n of a proper binary tree of height *h* is ?



nodes = 1	height = 0
nodes = 3	height = 1
nodes = 7	height = 2
nodes = $2^{h+1} - 1$	height = h

Since L = 2^h
and since the number of internal nodes = $2^{h}-1$ the total number of nodes n = $2^h + 2^{h} - 1 = 2(2^h) - 1 = 2^{h+1} - 1$.

If the number of nodes is n then what is the height?



nodes = 1	height = 0
nodes = 3	height = 1
nodes = 7	height = 2
nodes = n	height = $\text{Log}_2(n+1) - 1$

Since $n = 2^{h+1}-1$

$n + 1 = 2^{h+1}$

$\text{Log}_2(n+1) = \text{Log}_2 2^{h+1}$

$\text{Log}_2(n+1) = h+1$

$h = \text{Log}_2(n+1) - 1$

BinaryTree ADT

- The BinaryTree ADT extends the Tree ADT, i.e., it inherits all the methods of the Tree ADT
- Additional methods:
  - position leftChild(p)
  - position rightChild(p)
  - position sibling(p)

- Update methods may be defined by data structures implementing the BinaryTree ADT