The volume of high dimensional solids (2nd attempt)

Lecture notes for COL863, 25.01.2016

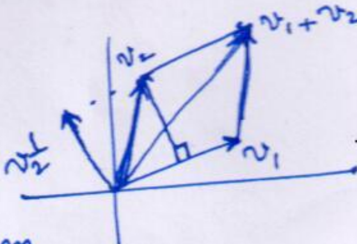
Definition: Given m linearly independent vectors $\{v_1, v_2 \dots v_m\}$ in $\mathbb{R}^n$ $(n \geq m)$, the m-parallelepiped defined by these vectors is

$$\{ y \in \mathbb{R}^n : y = c_1 v_1 + c_2 v_2 + \dots + c_m v_m ,\ 0 \leq c_i \leq 1, \forall i \}$$

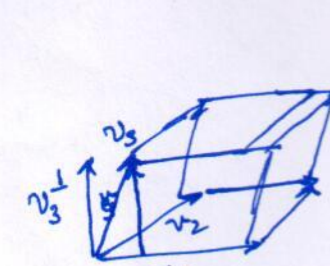
In these notes we may denote this PPD$(v_1, v_2, \dots v_m)$

The volume of a parallelepiped

Given a set of m vectors $v_1, v_2, \dots v_m \in \mathbb{R}^n$ $(n \geq m)$, we denote by $v_i^{\perp}$ the component of v_i that is orthogonal to the vectors $v_1, v_2 \dots, v_{i-1}$, or, we can say $v_i^{\perp}$ is the component of v_i that is orthogonal to the subspace spanned by the vectors $v_1, v_2, \dots, v_{i-1}$.



Let us now consider two vectors v_1 and v_2.
Note that the volume of the parallelepiped, what we usually call the area of the parallelogram defined by v_1 and v_2, is given by $\|v_1\| \|v_2^{\perp}\|$ (base × height).



Similarly, we can see ~~that the~~ volume of the solid body defined in 3 dimensions by v_1, v_2 & v_3 is $\|v_1\| \|v_2^{\perp}\| \|v_3^{\perp}\|$.

Definition The m-dimensional volume of the parallelepiped defined by the linearly independent vectors $v_1, v_2, \dots v_m \in \mathbb{R}^n$ is defined by

$$\|v_1\| \|v_2^{\perp}\| \|v_3^{\perp}\| \cdots \|v_m^{\perp}\|.$$

<u>Discussion</u>: Does the volume of the parallelepiped satisfy our intuition of what volume should be?

By "induction" it does. It satisfies our intuition in 2 & 3 dimensions as we have seen. Now suppose it satisfies our intuition in $m-1$ dimensions ie we believe that for vectors $v_1, v_2, \dots v_{m-1}$ the $m-1$ dimensional volume "should" be $\|v_1\| \|v_2^{\perp}\| \cdots \|v_{m-1}^{\perp}\|$.

Note that the parallelepiped defined by $v_1, v_2, \dots, v_{m-1}$ lies entirely in the subspace spanned by these $m-1$ vectors.

Now, to obtain the m dimensional volume of the parallelepiped defined by $v_1, \dots v_m$ let us consider slabs of infinitesimal thickness parallel to the PPD$(v_1, \dots v_{m-1})$

Translate the origin of these slabs along v_m.

If the thickness of these slabs is ϵ then the volume of each slab is $\text{Vol}(\text{PPD}(v_1, \dots v_{m-1})) \cdot \epsilon$

How many such slabs can there be? Exactly $\dfrac{\|v_m^{\perp}\|}{\epsilon}$.

And that gives us the volume of PPD$(v_1, \dots v_{m-1})$

$$= \lim_{\epsilon \to 0} \text{Vol}(\text{PPD}(v_1, \dots v_{m-1})) \cdot \epsilon \cdot \frac{\|v_m^{\perp}\|}{\epsilon}$$

$$= \|v_1\| \|v_2^{\perp}\| \cdots \|v_{m-1}^{\perp}\| \|v_m\|$$

Note: this is <u>not</u> a proof.

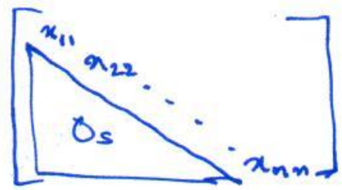
QR Factorization and the Gram Schmidt process

Any real square matrix A may be decomposed as

$$A = QR \text{ where}$$

Q is an orthogonal matrix (ie. its columns are orthogonal unit vectors $\Rightarrow Q^T Q = I$)

and R is an upper triangular matrix ie there are only 0s below the diagonal.



To see this for the special case of a square matrix A which is $n \times n$ such that the vectors comprising its columns are linearly independent, let us consider what is known as the Gram Schmidt process.

Let the columns of A be $a_1, \ldots a_n$. Denote by $\langle u, v \rangle$ the usual inner product defined on $\mathbb{R}^n$. Now define

$$u_1 = a_1 \qquad \text{and} \qquad e_1 = \frac{u_1}{\|u_1\|}$$

$$u_2 = a_2 - \frac{\langle a_2, u_1 \rangle}{\langle u_1, u_1 \rangle} . u_1 \qquad \text{and } e_2 = \frac{u_2}{\|u_2\|}$$

$$u_3 = a_3 - \frac{\langle a_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle a_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 \quad \text{and } e_3 = \frac{u_3}{\|u_3\|}$$

$$\vdots$$

$$u_k = a_k - \sum_{i=1}^{k-1} \frac{\langle a_k, u_i \rangle}{\langle u_i, u_i \rangle} u_i \qquad \text{and } e_k = \frac{u_k}{\|u_k\|}$$

till $k = n$.

From here note that we can rewrite our original vectors in terms of this orthonormal basis $e_1, \dots e_n$ as

$$a_1 = \langle e_1, a_1 \rangle e_1$$

$$a_2 = \langle e_1, a_2 \rangle e_1 + \langle e_2, a_2 \rangle e_2$$

$$\vdots$$

$$a_k = \sum_{j=1}^{k} \langle e_j, a_k \rangle e_j$$

noting that $\langle e_i, a_i \rangle = \|u_i\|$. This rewriting gives us that

$$A = QR$$

where $Q := [e_1, \dots e_n]$ is an ~~orthonormal mat~~ orthogonal matrix

and $R = \begin{pmatrix} \langle e_1, a_1 \rangle & \langle e_1, a_2 \rangle & \cdots & \langle e_1, a_n \rangle \\ 0 & \langle e_2, a_2 \rangle & & \vdots \\ & & \ddots & \vdots \\ 0 & & & \langle e_n, a_n \rangle \end{pmatrix}$

Now, returning to our definition of the volume of the parallelepiped $PPD(a_1, a_2 \dots a_n)$, note that the u_i defined above exactly satisfies the definition of $a_i^\perp$. Hence

$$Vol(PPD(a_1, a_2, \dots a_n)) = \|u_1\| \|u_2\| \dots \|u_n\|$$

(which is $\|a_1\| \|a_2^\perp\| \dots \|a_n^\perp\|$)

$$\Rightarrow Vol(PPD(a_1, a_2, \dots a_n)) = \langle e_1, a_1 \rangle \cdot \langle e_2, a_2 \rangle, \dots \langle e_n, a_n \rangle$$

$$= \det(R)$$

$$= \det(QR) \quad (\text{since } \det Q = 1)$$

$$= \det A.$$

Thm: Given vectors $v_1, v_2 \ldots v_m \in \mathbb{R}^n$ ($n \geq m$), the m-dimensional volume of the parallelepiped defined by these vectors is

$$\sqrt{\det(A^T A)}$$

where A is the $n \times m$ matrix with columns $v_1, v_2, \ldots v_m$.

In particular, when $m = n$, this volume is $|\det(A)|$.

The determinant as an "expansion" factor

Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Given a parallelepiped defined by vectors $v_1, v_2, \ldots v_m \in \mathbb{R}^n$, if we transform each ~~point~~ vector using T and consider the parallelepiped generated by those points then we can say that

$$T\left(PPD(v_1, v_2, \ldots v_m)\right) = PPD(Tv_1, Tv_2, \ldots Tv_m)$$

This follows from the linearity of T.

Hence if we want to find the volume of $PPD(v_1, v_2, \ldots v_m)$ under the transformation induced by T we can find the volume of $PPD(Tv_1, Tv_2, \ldots, Tv_m)$. ~~Hence~~ In other words,

$$\begin{aligned} \mathrm{Vol}\left(T(PPD(v_1, v_2, \ldots v_m))\right) &= \det\left[Tv_1, Tv_2, \ldots Tv_n\right] \\ &= |\det T| \, \mathrm{Vol}\left(PPD(v_1, v_2 \ldots v_n)\right). \end{aligned}$$

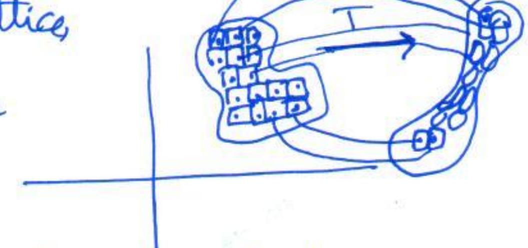
Thm: Given $v_1, v_2 \ldots v_n$ ~~linearly independent~~ vectors in $\mathbb{R}^n$, the n-dimensional volume of $PPD(v_1, v_2 \ldots v_n)$ under a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is equal to $|\det T| \, \mathrm{Vol}(PPD(v_1, \ldots v_n))$

Note: This can easily be extended to the case where we have an [illegible] in $\mathbb{R}^n$ ($n \geq m$) but we omit that here.

The determinant as expansion factor for arbitrary closed body volume

Thm: Given ~~an~~ a region Ω ~~with~~ in $\mathbb{R}^n$ and a linear transformation
$T: \mathbb{R}^n \to \mathbb{R}^n$, $\mathrm{Vol}\,(T(\Omega)) = |\det T|\ \mathrm{Vol}\,(\Omega)$.

Proof outline: If $\mathbb{Z}^n$ is the integer lattice, we denote by $\epsilon . \mathbb{Z}^n$ the points of the integer lattice scaled by ϵ.



Now, given a region Ω with non-zero n-dimensional volume, let $\mathrm{In}(\Omega) \subseteq \epsilon . \mathbb{Z}^n$ be the largest set of points of $\epsilon\mathbb{Z}^n$ such that $\bigcup_{x \in \mathrm{In}(\Omega)} B(x, \frac{\epsilon}{2})$ is completely contained in Ω where $B(x, \delta)$ is the ~~square~~ n-dimensional cube of side 2δ centred at x.

Similarly let $\mathrm{Out}(\Omega)$ be the smallest subset of $\epsilon\mathbb{Z}^n$ such that $\bigcup_{x \in \mathrm{Out}(\Omega)} B(x, \epsilon/2)$ completely contains Ω.

Since $B(x, \frac{\epsilon}{2})$ is a parallelepiped, the volume of $T(B(x, \frac{\epsilon}{2}))$ is given by $|\det T|\ \mathrm{Vol}\,(B(x, \frac{\epsilon}{2}))$

Noting that

$$\mathrm{Vol}\left(\bigcup_{x \in \mathrm{In}(\Omega)} B(x, \tfrac{\epsilon}{2}) \right) \leq \mathrm{Vol}\,(\Omega) \leq \mathrm{Vol}\left(\bigcup_{x \in \mathrm{Out}(\Omega)} B(x, \tfrac{\epsilon}{2}) \right)$$

~~using the linearity of T we~~ we get

$$\mathrm{Vol}\left(\bigcup_{x \in \mathrm{In}(\Omega)} |\det T|\ \mathrm{Vol}\,(B(x, \tfrac{\epsilon}{2})) \right) \leq |\det T|\ \mathrm{Vol}\,(\Omega) \leq \mathrm{Vol}\left(\bigcup_{x \in \mathrm{Out}(\Omega)} |\det T|\ \mathrm{Vol}\,(B(x, \tfrac{\epsilon}{2})) \right)$$

Noting that as $\epsilon \to 0$ the two bounds meet, their common limit

[illegible]

The volume of a simplex

Definition : Given $k+1$ points $u_0, u_1, \ldots u_k \in \mathbb{R}^{k+1}$ such that $u_1-u_0, u_2-u_0, \ldots u_k-u_0$ are linearly independent we say that the simplex determined by $u_0, u_1, \ldots u_k$ is the set

$$S_{k+1} = \left\{ \sum_{i=0}^{k} x_i u_i \ : \ \forall i: \ 0 \le x_i \le 1 \ , \ \sum x_i = 1 \right\}.$$

This is also known as a k-simplex. So, a 2-simplex is a triangle.

The "standard" k-simplex, also called the unit k-simplex in $\mathbb{R}^{k+1}$ is given by $u_0 = 0$, $u_1 = (1,0,\ldots 0)$, $u_2 = (0,1,0\ldots)$, $\ldots u_k = (0,0,\ldots 1)$.

Consider the following ~~to~~ linear transformation

$$T = \cancel{\begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}} \quad \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & & & & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

This maps the vertices of the unit k-simplex to $T(u_0) = (0,0,\ldots 0)$, $T(u_1) = (1,1,1, \ 1)$, $T(u_2) = (0,1,1,1\ldots 1)$, $T(u_3) = (0,0,1,1,\ldots 1)$, $T(u_k) = (0,0,\ldots,0,1)$.

In general it maps $(a_0, a_1, \ldots a_k) \in S$ to $(a_0, a_0+a_1, \ldots, \sum_{i=0}^{k} a_i)$.

If $R_{k+1} = T(S_{k+1})$ then

$$R_{k+1} = \{ \underline{x} : 0 \le x_0 \le x_1 \le \cdots x_k \le 1 \}$$

Note that since $\det(T) = 1$, $\mathrm{Vol}(R_{k+1}) = \mathrm{Vol}(S_{k+1})$.

We compute the volume of R by making a combinatorial argument.
Note that the coordinates of each point in R are sorted in ascending order (ie $x_0 \le x_1 \ldots x_k$) and $0 \le x_i \le 1 \ \forall \ 0 \le i \le k$.
The points of the unit cube can be partitioned into $(k+1)!$ partitions based on the different permutations describing the ordering of the coordinates. It is possible to rigorously argue (though we omit it here) that this implies that

$$\boxed{Vol\,(S_{k+1}) = \frac{1}{(k+1)!}}$$

<u>Volume of a general simplex</u>

Given a simplex $S_{k+1}(u_0, u_1, \ldots u_k)$ described by affinely indep points $u_0, u_1, \ldots u_k$ (ie points s.t. $u_1 - u_0, u_2 - u_0 \ldots u_k - u_0$ are linearly indep)
We transform it in 2 ways

1. Translate it to the origin by adding $-u_0$ to each point
2. Apply a linear transformation T such that
$$T(u_i - u_0) = (\underbrace{0, 0 \ldots}_{i-1}, 1, 0, \ldots 0)$$

i.e. $T(S_{k+1}(0, u_1 - u_0, \ldots u_k - u_0))$ = unit k-simplex

Now since $$Vol\,(T(S_{k+1}(0, u_1 - u_0, \ldots, u_k - u_0))) = \frac{1}{(k+1)!}$$

we have that $$Vol\,(S_{k+1}(u_0, u_1, \ldots u_k)) = \frac{|\det T^{-1}|}{(k+1)!}$$

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